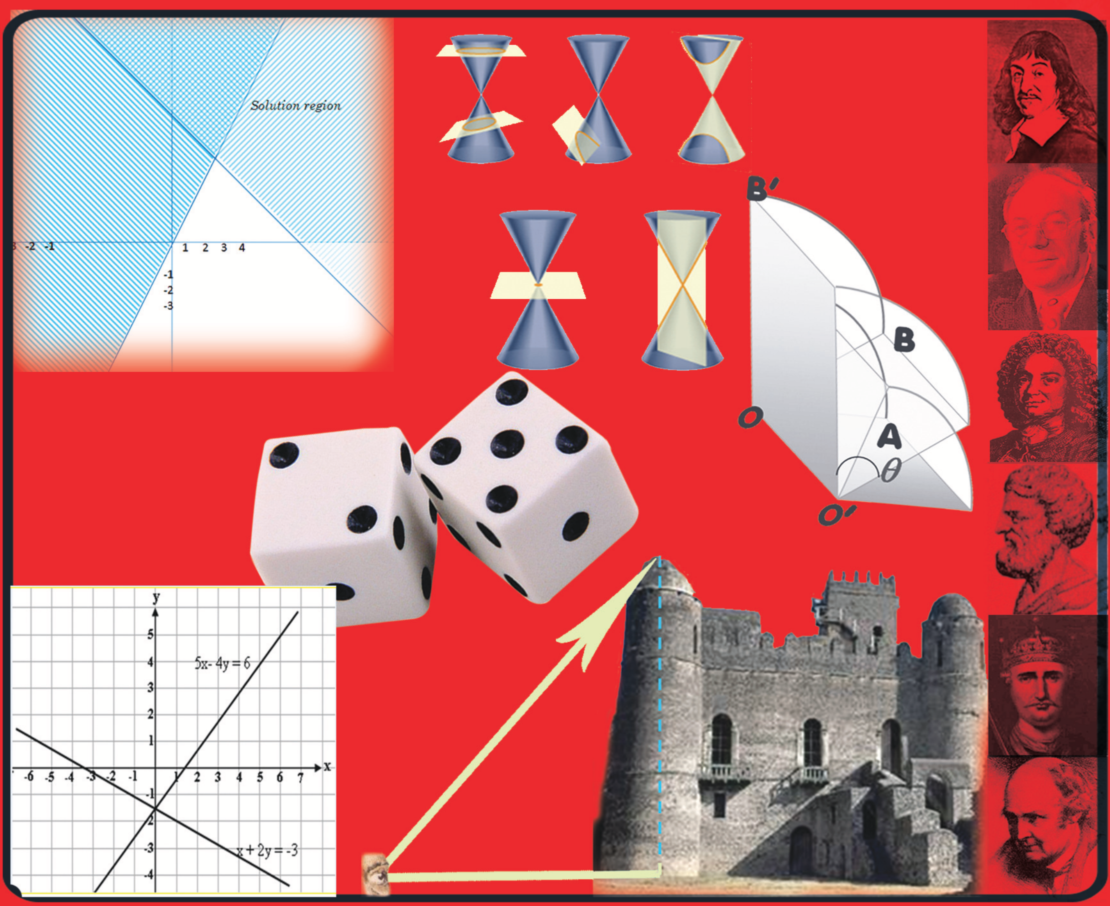
MATHEMATICS

STUDENT TEXTBOOK GRADE 11



FEDERAL DEMOCRATIC REPUBLIC OF ETHIOPIA
MINISTRY OF EDUCATION

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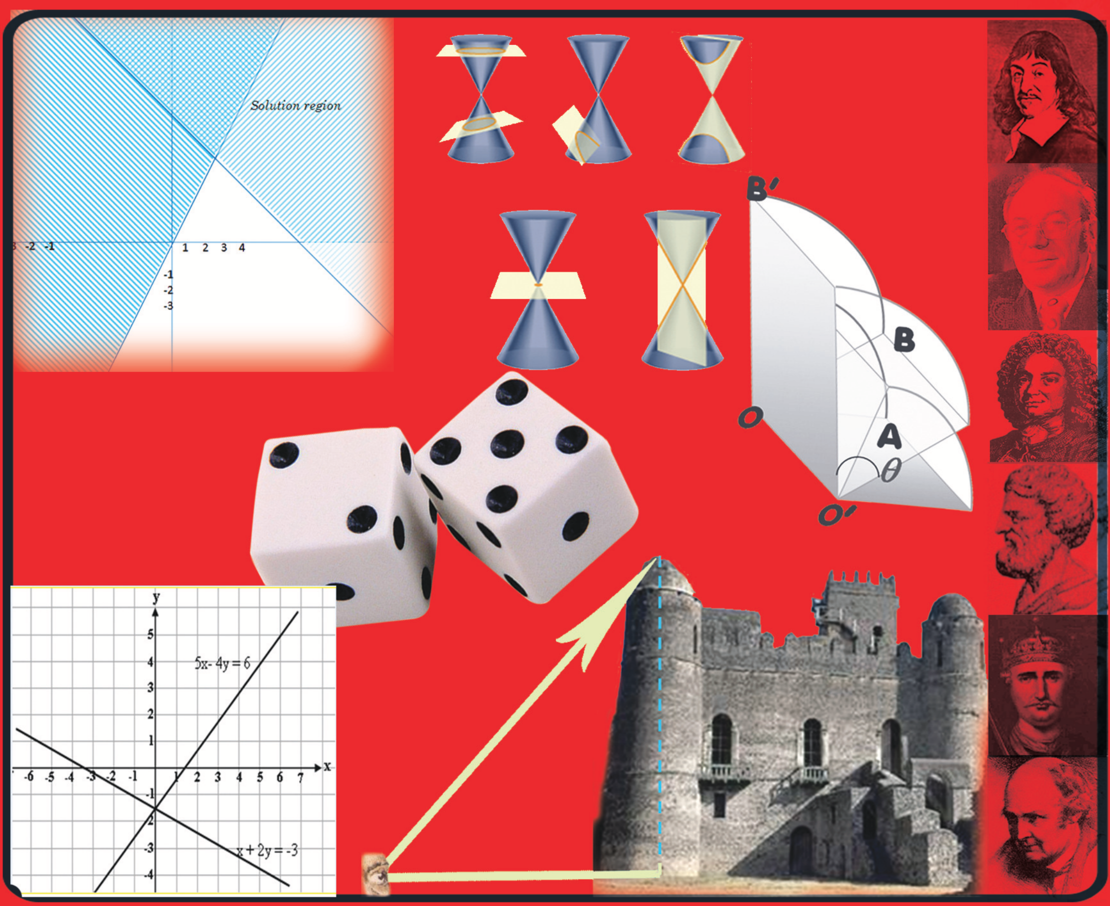

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STUDENT TEXTBOOK

GRADE11

Authors, Editors and Reviewers:

Rachel Mary Z. (B.Sc.)

Kassa Michael (M.Sc.)

Kinfegabrail Dessalegn (M.Sc.)

Sebsibe Teferi (M.Sc.)

Tilahun Abebaw (Ph.D.)

Hunduma Legesse (M.Sc.)

Mulugeta Naizghi (M.Sc.)

Birhanu Bekele (Ph.D)

Evaluators:

Tesfaye Ayele

Dagnachew Yalew

Tekeste Woldetensai

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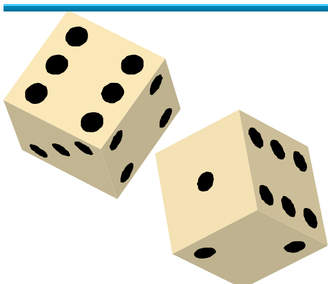

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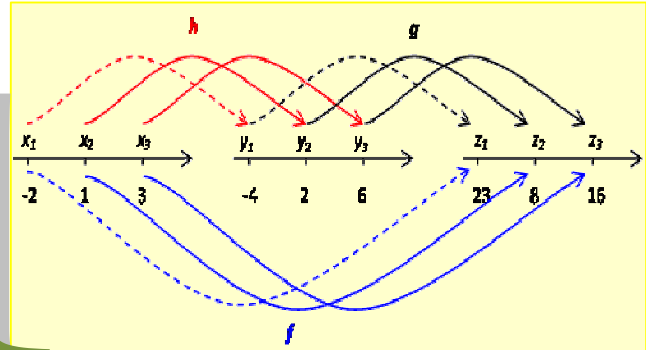

Unit 1

FURTHER ON RELATIONS AND FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *know specific facts about relations.*
- *know additional concepts and facts about functions.*
- *understand methods and principles in composing functions.*

Main Contents

1.1 REVISION ON RELATIONS

1.2 SOME ADDITIONAL TYPES OF FUNCTIONS

1.3 CLASSIFICATION OF FUNCTIONS

1.4 COMPOSITION OF FUNCTIONS

1.5 INVERSE FUNCTIONS AND THEIR GRAPHS

Key terms

Summary

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INTRODUCTION

Relationships between elements of sets occur in many contexts. Examples of relations in society include one person being a brother of another person or one person being an employee of another.

On the other hand, in a set of numbers, one number being a divisor of another, or one number being greater than another are some examples of relations.

In Grades 9 and 10, you learned a great deal about relations and functions. In this unit you will study some more about them. We hope that your understanding of the concepts will be strengthened. You will also study some additional types of functions.

HISTORICAL NOTE

Rene Descartes (1596 - 1650)

Rene Descartes was a philosopher and a mathematician, who assigned coordinates to describe points in a plane. The *xy*-coordinate plane is sometimes called the Cartesian plane in honour of this Frenchman. Descartes' discovery of the Cartesian coordinate system helped the growth of mathematical discoveries for more than 200 years.

John Stuart Mill called Descartes' invention of the Cartesian plane "*The greatest single step ever made in the progress of the exact sciences*".

OPENING PROBLEM

A set of glasses that are in the shape of right circular cones are to be made for display as shown in the adjacent figure. The glasses have the same height, h = 50cm. If the volume of a cone v, as a function of r, is given by the formula $v = f(r) = \frac{1}{3}\pi h r^2$

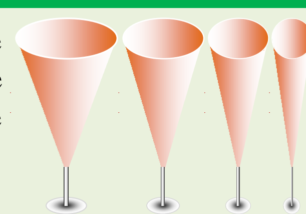


Figure 1.1

- Can you express $r = f^{-1}(v)$?
- Can you fill in the following table? Measurements are in cm. Round r to two decimal places

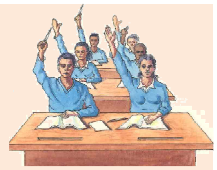
v	40	80	120	160	200	240	280
r							

- Can you draw the graph of f^{-1}?

1.1 REVISION ON RELATIONS

1.1.1 Inverse of a Relation

ACTIVITY 1.1



1 Let $A = \{1, 5, 6, 7, 8\}$ and $B = \{-1, 2, 4, 9\}$ be two sets and

$R = \{(5, -1), (6, 4), (7, 9), (8, 2), (1, -1)\}$ be a relation from A to B.

Give the domain and the range of R.

2 Let $R = \{ (x, y) : x < y\}$. Which of the following ordered pairs belong to R?

a $(-5, 6)$ **b** $(\pi, 3.4)$ **c** $(-4, -6.234)$

3 Reverse the order of each of the ordered pairs in R in Questions 1 and 2 above.

Note:

- ✓ A relation is a set of ordered pairs.
- ✓ Given two sets A and B, a relation from A to B is defined as any subset of $A \times B$.
- ✓ A relation on A is any subset of $A \times A$.
- ✓ Let R be a relation from A to B. Then,

  Domain of $R = \{x \in A: (x, y) \in R, \text{ for some } y \in B\}$

  Range of $R = \{y \in B: (x, y) \in R, \text{ for some } x \in A\}$

If R is a relation from A to B, then you may want to know what the inverse of R is. The following definition explains what we mean by the inverse of a relation.

Definition 1.1

Let R be a relation from A to B. The inverse of R, denoted by R^{-1}, is a relation from B to A, given by

$$R^{-1} = \{(b, a): (a, b) \in R\}.$$

Example 1 Let $A = \{0, -1, 2\}$ and $B = \{5, 6\}$.

Give the inverse of $R = \{(0, 5), (0, 6), (-1, 6)\}$.

Solution $(a, b) \in R$ means $(b, a) \in R^{-1}$. Thus, $R^{-1} = \{(5, 0), (6, 0), (6, -1)\}$

Example 2 Let A be the set of all towns in Ethiopia, and B be the set of all regions in Ethiopia. If R = {(*a*, *b*): town *a* is found in region *b*}, then find R^{-1}.

Solution Notice that, the 1st element of any ordered pair in R is a town, while the 2nd element is a region.

Thus, in R^{-1}, the 1st element of the ordered pair should be a region while the 2nd element should be a town.

So, $R^{-1} = \{(b, a)$: region b contains town $a\}$

$= \{(a, b)$: region a contains town $b\}$

Example 3 Let $R = \{(x, y) : y = x + 3\}$. Find R^{-1}.

Solution In R, the 2nd coordinate is 3 plus the 1st coordinate. Thus,

$$R^{-1} = \{(y, x) : (x, y) \in R\} = \{(x, y) : (y, x) \in R\}.$$

$= \{(x, y) : x = y + 3\}$. Notice that the 1st coordinate is 3 plus the 2nd coordinate.

$= \{(x, y) : y = x - 3\}$. Solve for *y*.

Example 4 Let $R = \{(x, y): y \le x + 3 \text{ and } y > -2x + 6\}$. Give R^{-1}.

Solution $R^{-1} = \{(y, x): y \le x + 3 \text{ and } y > -2x + 6\}$

$= \{(x, y): x \le y + 3 \text{ and } x > -2y + 6\}$

$$= \left\{(x, y) : y \ge x - 3 \text{ and } y > -\frac{1}{2}x + 3\right\}$$

Group work 1.1

1 If A = {1, 2, 3, 4, 5} and B = {*u*, *v*, *w*, *x*}, then which of the following are relations from A to B?

- **a** $R_1 = \{(1, v), (2, w), (5, x)\}$
- **b** $R_2 = \{(1, v), (3, 3), (4, v), (4, w)\}$
- **c** $R_3 = \{(1, y), (1, x), (3, v), (3, x)\}$
- **d** $R_4 = \varnothing$

2 For the relation in Example 1 above,

- **a** Find the domain and range of R.
- **b** Find the domain and range of R^{-1}.
- **c** Compare the domain of R with the range of R^{-1} and the range of R with the domain of R^{-1}. What do you notice?

3 For the relation given in Example 2 on the previous page, if Ambo town is in Oromia region and Jijiga town is in Somali region, which of the following is in R^{-1}?

a (Jijiga, Somali) **b** (Oromia, Jijiga)

c (Oromia, Ambo) **d** (Somali, Jijiga)

4 For the relation in Example 4 on the previous page, find the domain and range of R^{-1}.

5 Give the domain and range of the inverse of each of the following relations.

a $R = \left\{(1,5), (3, -6), (4, 3.5), \left(1, \frac{6}{5}\right)\right\}$

b $R = \{(x, y) : y = 3x - 7\}$

c $R = \{(x, y) : y < -3x \text{ and } y \geq x - 4\}$

Exercise 1.1

1 If $R = \{(x, y): y \geq x + 1\}$, which of the following is true?

a $(0, 0) \in R$ **b** $0 \in$ Domain of R

c $(0, 1) \in R$ **d** $(-5, 6) \in R$

e $(-5, -5) \in R$ **f** $0 \in$ Range of R.

2 Let $R = \{ (x, y) : y \geq x^2 - 1 \text{ and } y \leq 3 \}$

a Sketch the graph of R.

b Give the domain and the range of R.

3 Give the relation represented by the shaded region in Figure 1.2.

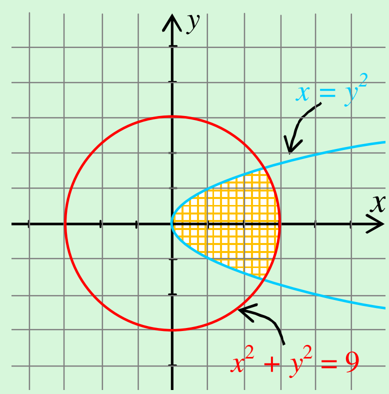


Figure 1.2

4 Give the inverse of each of the following relations.

a $R = \{ (x, y): x \text{ is a brother of } y\}$

b $R = \{ (x, y) : x^2 + 1 = y^2 \}$

c $R = \{ (x, y) : y \geq x + 3 \text{ and } y < -3x - 1\}$

5 Give the domain and range of the inverse of each of the following relations.

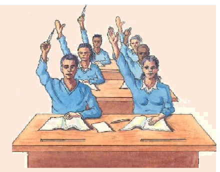
a $R = \{ (x, y) : y \geq x^2 + 1\}$

b $R = \{ (x, y) : y \leq -x^2 \text{ and } y \geq -1\}$

c $R = \{ (x, y) : -3 \leq x \leq 3, y \in \mathbb{R}\}$

1.1.2 Graphs of Inverse Relations

ACTIVITY 1.2



Do the following in pairs.

Let R = {(1, –2), (3, 9), (4, 6), (5, –7), (5, 2.5)}

- **a** List the elements of R^{-1}.
- **b** Compare the domain of R^{-1} and the range of R. What do you notice?
- **c** Compare the range of R^{-1} and the domain of R. What do you notice?
- **d** Do the same for $R = \{(x, y): -3 \le x \le 3, y \in \mathbb{R}\}$.
- **e** How can you generalize your findings?

From what you did so far, you should have concluded that

Domain of R^{-1} = Range of R

Range of R^{-1} = Domain of R

Note:

✓ On the Cartesian coordinate plane, in keeping with common usage, arrows are used on the axes to show positive direction.

✓ If the boundary curve in the graph of a relation is not part of the relation, it is shown using a broken line.

Now, let us compare graphs of R and R^{-1} and see their relationship.

Example 5 Let $R = \{(x, y): y \ge x^2\}$. Draw the graph of R and R^{-1} using the same coordinate axes.

Solution $R^{-1} = \{(x, y): x \ge y^2\}$.

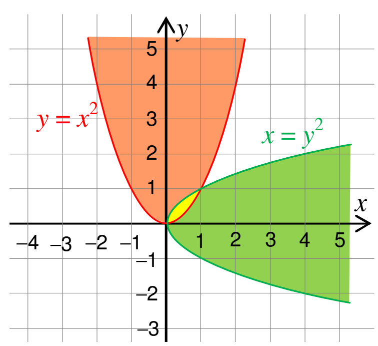


Figure 1.3 *The graph of R and R^{-1}.*

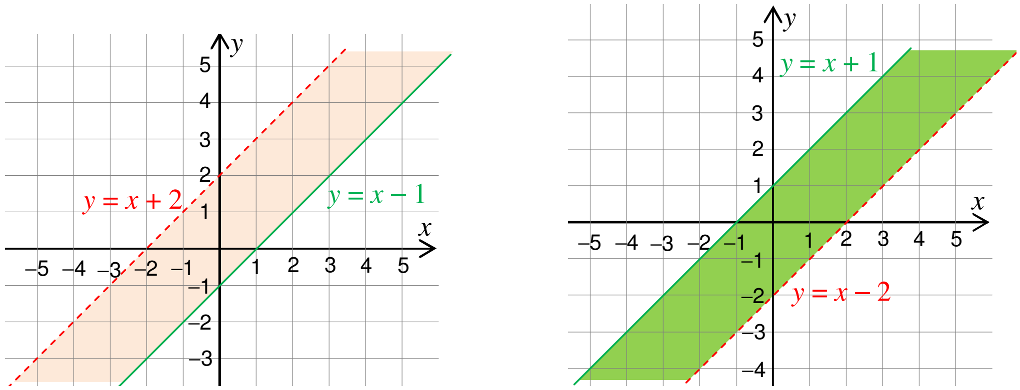
Notice that $y = x^2$ and $x = y^2$ meet at (0, 0) and (1, 1). The equation of the line through the two points is $y = x$.

Example 6 For the following relation, sketch the graphs of R and R^{-1} using different coordinate axes.

$$R = \{(x, y): y < x + 2 \text{ and } y \geq x - 1\}.$$

Solution $R^{-1} = \{(x, y): x < y + 2 \text{ and } x \geq y - 1\}$

$= \{(x, y): y > x - 2 \text{ and } y \leq x + 1\}$



a *The graph of R* **b** *The graph of R^{-1}*

Figure 1.4

Group Work 1.2

1 Let R = {(3, –1), (4, 2), (6, 3), (–5, 1)}

  a List the elements of R^{-1}

  b On a piece of squared paper, sketch the line $y = x$.

  c Sketch R and R^{-1} on the paper, using different colours (or marking points of R by * and points of R^{-1} by Δ).

  d Fold the paper along the line $y = x$.

  e What do you notice?

2 Let $R = \{(x, y): y = x^3\}$. Give R^{-1}. Repeat the above investigation b to e for R.

3 Sketch the graph of $R = \{(x, y): y < x + 2 \text{ and } y \geq x - 1\}$ on squared paper; then turn the paper over, rotate it 90° clockwise, and finally hold it up to the light. What do you see through the paper? Compare it with the graph of R^{-1} in Example 6 above. Why does this procedure work?

From the above Group Work, you should conclude that R and R^{-1} are mirror images of each other on the line $y = x$. This means, if you reflect the graph of R in the line $y = x$, you get the graph of R^{-1} and vice versa.

Exercise 1.2

1 **a** Let R = $\{(x, y): x + 1 = y^2\}$. Draw the graph of R^{-1} by reflecting the graph of R in the line $y = x$.

b Consider the following graph of a relation R.

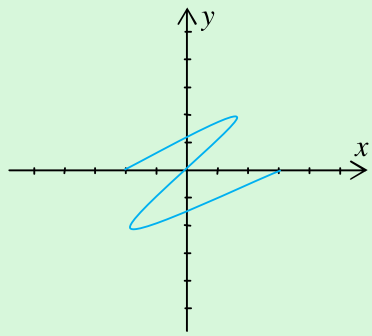
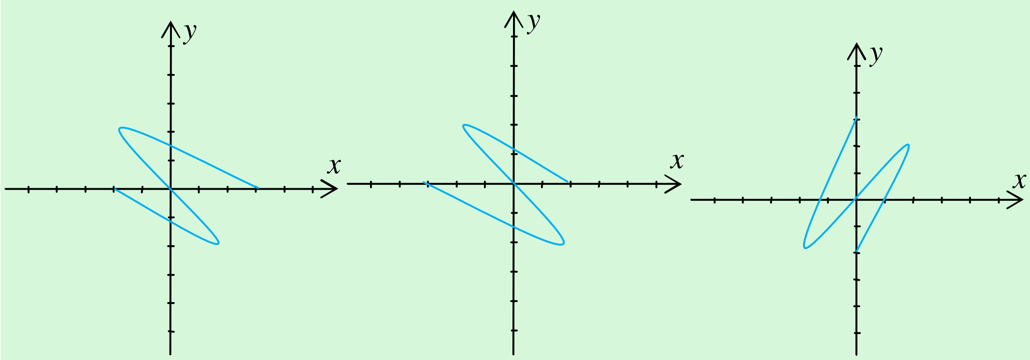


Figure 1.5

Which of the following is the graph of the inverse of R?



i ii iii

Figure 1.6

2 For each of the following relations, draw the graph of R and its inverse using the same coordinate axes.

a R = $\{ (x, y) : x + y \le 1\}$

b R = $\{ (x, y) : y \ge x + 1$ and $y < -3x\}$

c R = $\{ (x, y) : x^2 + y^2 = 16\}$

d R = $\{ (x, y) : x^2 + y^2 > 16\}$

1.2 SOME ADDITIONAL TYPES OF FUNCTIONS

1.2.1 Revision on Functions

ACTIVITY 1.3

1 Which of the following are functions?

a $R = \{(1,0),(2,0),(3,1),(1,6)\}$ b $S = \{(1,0),(2,0),(3,1),(6,1)\}$

c $T = \{(x, y): y = 2x-1\}$ d $W = \{(x, y): y \geq 2x + 9\}$

e $K = \{(1,3),(3,2),(1,7),(-1,4)\}$ f $L = \{(0,0),(0,-2),(0,2),(0,4)\}$

2 Let $f(x) = 9x - 2$ and $g(x) = \sqrt{3x+7}$. Find the domains of f and g and evaluate the following:

a $f(-2)$ b $f\left(\frac{-1}{2}\right)$ c $g(3)$

Note:

- ✓ A function is a relation in which no two distinct ordered pairs have the same first element.
- ✓ If f is a function with domain A and range a subset of B, we write
  $f: A \rightarrow B$ or $A \xrightarrow{f} B$
- ✓ If $f: A \rightarrow B$ is given by a rule that maps x from A to y in B, then we write $y = f(x)$.

Example 1 Suppose f: A $\rightarrow$ B is the function that gives $5x - 1$ for any $x \in$ A. What are the possible ways of writing this function?

Solution We can write it as

$$f: x \rightarrow 5x - 1 \text{ or } f(x) = 5x - 1 \text{ or } y = 5x - 1 \text{ or } x \xrightarrow{f} 5x - 1.$$

Note:

- ✓ $f(x)$ is read as " f of x ".
- ✓ $y = f(x)$, if and only if $P(x, y)$ is a point on the graph of f.

Vertical line test:

A set of points in the Cartesian plane is the graph of a function, if and only if no vertical line intersects the set more than once.

Definition 1.2

A function $f: A \to B$ is said to be

i odd, if and only if, for any $x \in A$, $f(-x) = -f(x)$.

ii even, if and only if, for any $x \in A$, $f(-x) = f(x)$. The evenness or oddness of a function is called its **parity**.

Example 2

a $f(x) = x^3$ is odd, since $f(-x) = (-x)^3 = -x^3 = -f(x)$.

b $f(x) = x^2$ is even since $f(-x) = (-x)^2 = x^2 = f(x)$.

c $f(x) = x + 1$ is neither even nor odd since $f(-x) = -x + 1 \neq -(x + 1) = -f(x)$ and $f(-x) = -x + 1 \neq x + 1 = f(x)$,

✍Note:

Exponential and Logarithmic Functions

✓ A function $f: \mathbb{R} \to (0,\infty)$ given by $f(x) = a^x$, $a > 0$, $a \neq 1$ is called an **exponential function**.

✓ A function $f: (0,\infty) \to \mathbb{R}$ given by $f(x) = \log_a x$, $a > 0$, $a \neq 1$ is called a **logarithmic function**.

✓ If $a > 0$, $a \neq 1$, then, $\log_a a^x = a^{\log_a x} = x$.

Exercise 1.3

1 Draw the graph of each of the following functions:

a $f(x) = \dfrac{3x-1}{2}$ **b** $g(x) = \sqrt{x+1}$ **c** $f(x) = 4$

2 A researcher investigating the effect of pollution on plant life found that the percentage $p(x)$ of diseased trees and shrubs at a distance of x km from an industrial city is given by $p(x) = 32 - \dfrac{3x}{50}$, for $50 \leq x \leq 500$. Sketch the graph of the function p and find $p(50)$, $p(100)$, $p(200)$, $p(400)$.

3 Determine whether each of the following functions is even, odd or neither.

a $g(x) = \sqrt{8x^4 + 1}$ **b** $f(x) = 4x^3 - 5x$

c $f(x) = x^4 + 3x^2$ **d** $h(x) = \dfrac{1}{x}$

4 Use the vertical line test to determine the graph(s) that represent function(s).

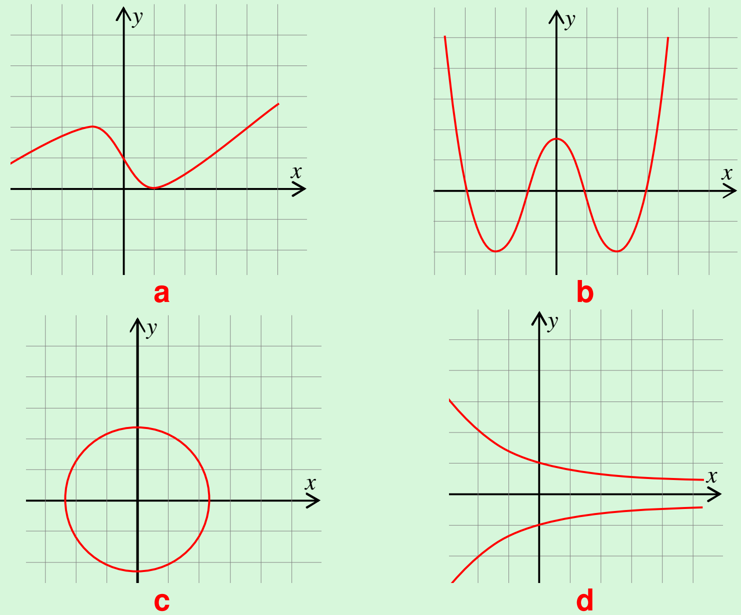
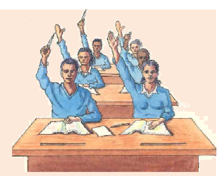


Figure 1.7

1.2.2 Power Functions



ACTIVITY 1.4

Given the functions

a $f(x) = 2x^7$ **b** $g(x) = x$ **c** $h(x) = 4^x$

d $f(x) = x^{\frac{3}{2}}$ **e** $g(x) = \left(\frac{2}{3}\right)^x$

Classify each as a power function or an exponential function.

Definition 1.3

A power function is a function which can be written in the form $f(x) = ax^r$, where r is a rational number and a in $\mathbb{R}$, is a fixed number.

✍Note:

Don't confuse power functions with exponential functions.

Exponential function: $y = a^x$ (a fixed base is raised to a variable exponent)

Power function: $y = ax^r$ (variable base is raised to a fixed exponent)

Let us see the behaviour of a power function when r is an integer.

Group Work 1.3

Do the following in groups.

i When r is a positive integer

1 Let $f(x) = 4x^3$

a What is the domain of f? What is the range of f?

b Fill in the following table.

x	–2	–1	0	1	2
$f(x)$					

c Sketch the graph of $f(x)$ using the above table.

d What is the parity of f (i.e. is it even or odd)?

e Investigate its symmetry.

2 Go through the steps a to e for the function $f(x) = 4x^2$

ii When r is a negative integer

3 Let $f(x) = 2x^{-3}$

a What is the domain of f? What is the range of f?

b Fill in the following table.

x	–2	–1	0	1	2
$f(x)$					

c Sketch the graph of $f(x)$ using the above table.

d What is the parity of f (i.e. is it even or odd)?

e Investigate its symmetry.

4 Go through the steps a to e for the function $f(x) = 2x^{-2}$.

We now consider the behaviour of a power function when r is a rational number of the form $\frac{m}{n}$, where m and n are integers, with $n \neq 0$. (We will assume that $\frac{m}{n}$ is in its lowest term.)

Example 3 Draw the graph of $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$.

Solution The following table gives some values.

x	–8	–1	0	1	8
$f(x)$	–2	–1	0	1	2

Using the above values, the graph can be sketched as:

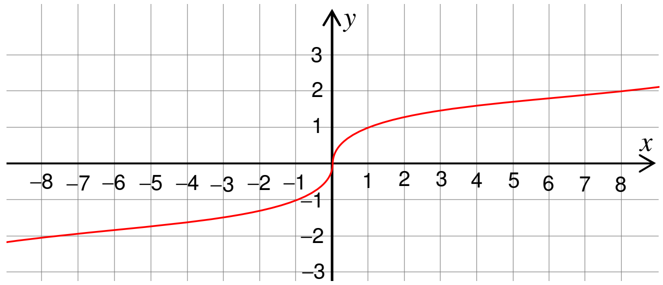


Figure 1.8 *Graph of* $f(x)=\sqrt[3]{x}$.

Note:

1 For the function $f(x) = \sqrt[3]{x}$, Domain of f = Range of $f = \mathbb{R}$.

2 The point (0, 0), where the graph changes shape from concave upward to concave downward, is called an **inflection point**.

3 All functions $f(x) = x^{\frac{1}{n}}$, where n is an odd natural number, have similar behaviours. They all pass through (–1, –1) and (1, 1). They are also increasing.

The following figures give you some of the various possible graphs of power functions with rational exponents.

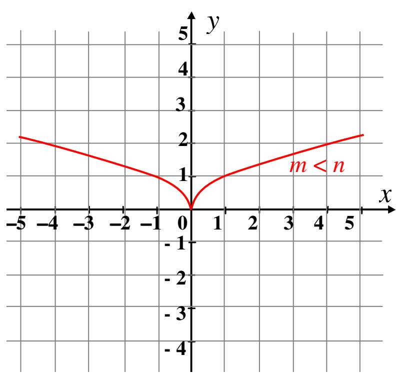


Figure 1.9 *Type of graph of* $y = x^{\frac{m}{n}}$, *m even, n odd*

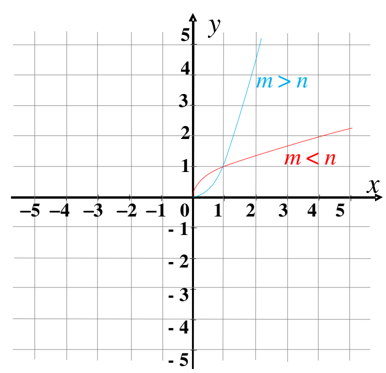


Figure 1.10 *Type of graph of* $y = x^{\frac{m}{n}}$, *m odd, n even*

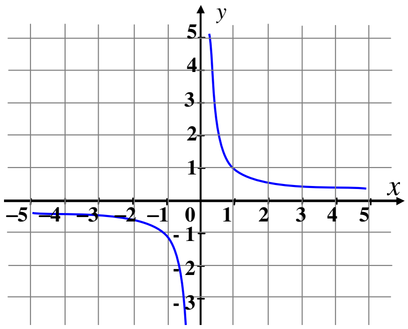


Figure 1.11 *Graph of* $y = x^{\frac{-1}{n}}$, n odd

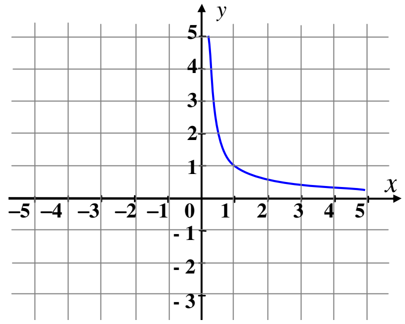


Figure 1.12 *Graph of* $y = x^{\frac{-1}{n}}$, n even

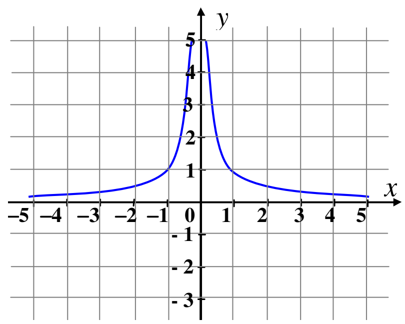
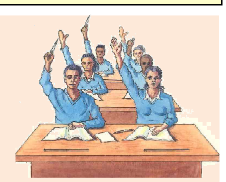


Figure 1.13 *Graph of* $y = x^{\frac{-m}{n}}$ *, m even, n odd*

Note:

The point P(0, 0) in Figure 1.9 is called a **cusp**.

ACTIVITY 1.5



Answer each of the following for the functions given by Figures 1.9-1.13 above.

1. What are their domains and ranges?
2. Give their parities.
3. State whether they are symmetrical with respect to the *x*-axis, *y*-axis, the origin or neither.
4. Where are they increasing and where are they decreasing?

Exercise 1.4

1. Which of the following are power functions and which are not? (Give reasons).

a $f(x) = 5x^2 + 1$ **b** $f(x) = 5x^{\frac{-3}{4}}$ **c** $g(x) = x^{-2}$

d $h(x) = x^x$ **e** $l(x) = 5^{x+1}$

2 Find the domain of each of the following power functions.

a $f(x) = x^{\frac{1}{3}}$ **b** $f(x) = x^{\frac{5}{4}}$ **c** $f(x) = 2x^{\frac{-2}{3}}$ **d** $f(x) = x^{\frac{-7}{4}}$

3 Sketch the graphs of f, g and h using the same coordinate axes.

$f(x) = x^2$, $g(x) = 2x^2$ and $h(x) = -2x^2$.

4 If $f(x) = ax^n$, $a \neq 0$ and $f(xy) = f(x)\,f(y)$, what is the value of a?

5 Consider $f(x) = ax^{-1}$, $a \neq 0$.

a Give the domain and range of $f(x)$.

b Suppose $a > 0$. Then $y = f(x)$ can be written as $y = \frac{a}{x}$ or $xy = a$. Here x and y are inversely related and a is called the **constant of variation**. Draw the graph of $f(x)$, when $a = 2$, and describe its symmetry.

1.2.3 Absolute Value (Modulus) Function

ACTIVITY 1.6

Find the absolute value of each of the following.

a -2 **b** 3 **c** 0 **d** -6.014

Definition 1.4

For any real number a, the **absolute value** or **modulus** of a, is defined by

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Calculator Tips

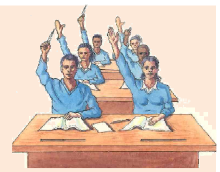
Some calculators have keys denoted by $|x|$ or ABS.

You can use such a key to find the absolute value of a number.

In case you have a calculator that does not have such a key, to find $|x|$, enter x, press the x^2 key, and then press $\sqrt{\ }$ key.

This is based on the property $|x| = \sqrt{x^2}$

ACTIVITY 1.7



1 Compare the absolute values of

a -3.5 and 3.5 **b** 4.213 and -4.213

c What can you conclude about $|x|$ and $|-x|$, for any x in $\mathbb{R}$?

2 **a** Compare $|xy|$ and $|x||y|$ for the following.

i $x = 2.4, y = 3$ **ii** $x = -6, \; y = 4$

b Conclude whether or not $|xy| = |x||y|$, for all $x, y \in \mathbb{R}$

Some properties of the absolute value

1 $|x| \geq 0$ for any $x \in \mathbb{R}$.

2 $|x|$ is the distance between the point corresponding to x and the origin.

3 $|x| \geq x$ and $|x| \geq -x$, for any point with coordinate x in $\mathbb{R}$.

4 $|x| = |-x|$, for any $x \in \mathbb{R}$

5 For any $x, y \in \mathbb{R}$, $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, provided that $y \neq 0$

6 $|xy| = |x||y|$, for any $x, y \in \mathbb{R}$

7 $|x| = a$, if and only if $x = \pm a$, provided, $a \geq 0$. In case $a < 0$, then $|x| = a$ has no solution.

Definition 1.5

The **modulus** (**Absolute value**) function is the function given by $f(x) = |x|$.

✍Note:

Domain of $f = \mathbb{R}$. Since $f(x) = |x| \geq 0$, for each $x \in \mathbb{R}$, Range of $f = [0, \infty)$.

Example 4

a Complete the following table for $f(x) = |x|$.

x	–3	–2	–1	0	1	2	3
$f(x)$							

b Using the above table, sketch the graph of $f(x) = |x|$ and notice its features.

Solution

a

x	–3	–2	–1	0	1	2	3
$f(x)$	3	2	1	0	1	2	3

b From the table, you can draw the graph as follows:

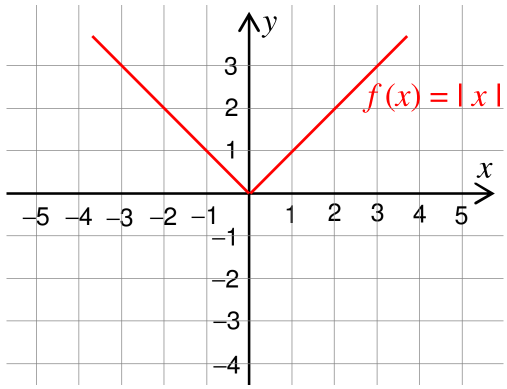


Figure 1.14 *Graph of* $y = |x|$.

As you can see, the graph has no hole or break in it (i.e. it is continuous) and it makes a sharp corner or a cusp at P(0, 0). The graph is also symmetrical with respect to the y-axis.

Exercise 1.5

1 If $x = 4$ and $y = –6$, then find:

a $|4x – 3|$ **b** $|xy| + 1$ **c** $\frac{|x|}{x+1}$

2 Give the solution sets for each of the following equations.

a $|x| = 4$ **c** $|3x + 1| = 0$

b $|x – 3| = –1$ **d** $|3x + 1| = 5$

3 Give the domain of each of the following functions.

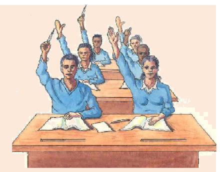
a $f(x) = |x| + 1$ **c** $h(x) = \left|\frac{1}{x}\right|$

b $g(x) = |x| – x$ **d** $k(x) = x – \left|\frac{x}{2}\right|$

4 Sketch the graphs of $f(x)$ and $h(x)$ given in Question 3 above.

1.2.4 Signum Function

ACTIVITY 1.8



Consider the function $f(x) = \begin{cases} 3, & \text{if } x \geq 0 \\ -2, & \text{if } x < 0 \end{cases}$. Find

a the domain of f **b** the range of f

c Sketch the graph of f.

Definition 1.6

The **signum function**, read as signum x, is written as sgn x and is defined by

$$y = f(x) = \text{sgn } x = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$

Since, $\dfrac{|x|}{x} = \begin{cases} 1, & x > 0 \\ \nexists \text{(does not exist)}, & x = 0 \\ -1, & x < 0 \end{cases}$, we have, $\text{sgn } x = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Note:

✓ The symbol $\nexists$ means does not exist or undefined.

✓ The signum function is an example of a **piecewise-defined** function.

✓ If an end point of a curve is not part of the graph, it is shown by a small open circle (⊸).

✓ If an end point of a curve is part of the graph, it is shown by a small filled - in circle (—•).

Example 5

a Complete the following table.

x	–4	–3	–2	–1	0	1	2	3	4
sgn x									

b Sketch the graph of $f(x)$ = sgn x using the above table and find its domain and range .

Solution

a

x	–4	–3	–2	–1	0	1	2	3	4
sgn x	–1	–1	–1	–1	0	1	1	1	1

b

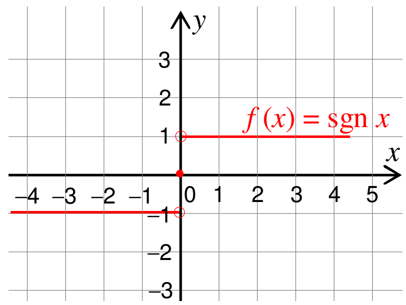


Figure 1.15

As you can see from the graph, the domain of $f(x) = \text{sgn } x$ is ℝ and its range is $\{-1, 0, 1\}$.

Exercise 1.6

1 Sketch the graph of $f(x) = x + \text{sgn } x$. Give the domain and range of f.

2 Draw the graph of $f(x) = x \text{ sgn } x$. What is its relationship with the graph of $y = |x|$?

3 Sketch the graph of $f(x) = x^2 \text{ sgn } x$. What is its domain? What is its range?

4 Sketch the graph of $g(x) = x^3 \text{ sgn } x$. Give its domain and range. Does it have symmetry with respect to any line?

5 **a** Is $f(x) = \text{sgn } x$ even or odd? **b** Is $f(x) = x^3 \text{ sgn } x$ even or odd?

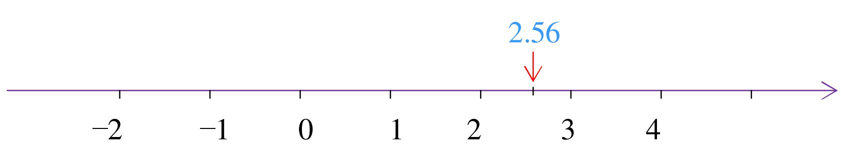
1.2.5 Greatest Integer (Floor) Function

Definition 1.7

The greatest integer function, denoted by

$f(x) = \lfloor x \rfloor$, is defined as the **greatest integer** $\leq x$.

Example 6 Let $x = 2.56$. On the number line, x is found between 2 and 3.



What is the largest among the integers that is less than or equal to 2.56?

You can see that it is 2.

Thus, $\lfloor 2.56 \rfloor = 2$.

Note:

The greatest integer $\leq x$ is also called the **floor** of x.

Example 7 Find $\lfloor x \rfloor$ when

a $x = -4.6$ **b** $x = 3$ **c** $x = 7.2143\ldots$

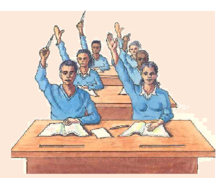
Solution

a -5 is the largest integer ≤ -4.6. i.e., $\lfloor -4.6 \rfloor = -5$.

b 3 is the largest integer ≤ 3. i.e. $\lfloor 3 \rfloor = 3$.

c $7.2143\ldots$ is between 7 and 8. Thus, $\lfloor 7.2143 \rfloor = 7$.

ACTIVITY 1.9



1 Let $f(x) = \lfloor x \rfloor$.

a Give $f(-3)$, $f(-2.7)$, $f(-2.5)$, $f(-2.1)$, $f(-2.01)$

b What is $f(x)$, when $-3 \leq x < -2$?

c Complete the following table.

x	$-3 \leq x < -2$	$-2 \leq x < -1$	$-1 \leq x < 0$	$0 \leq x < 1$	$1 \leq x < 2$	$2 \leq x < 3$
$f(x)$	-3	-2				2

2 Draw the graph of $f(x) = \lfloor x \rfloor$.

Note:

The greatest integer $\leq x$ is the integer that is immediately to the left of x (or x itself, if x is an integer).

As you have seen from the examples above, x can be any real number but $\lfloor x \rfloor$ is always an integer. Thus, Domain = $\mathbb{R}$; Range = $\mathbb{Z}$

We write this as $f: \mathbb{R} \rightarrow \mathbb{Z}$ given by $f(x) = \lfloor x \rfloor$.

Exercise 1.7

1 What is the value of each of the following?

a $\lfloor \pi \rfloor$ b $\lfloor -21.01 \rfloor$ c $\lfloor 21.01 \rfloor$ d $\lfloor 0 \rfloor$

2 Given $f(x) = \lfloor x \rfloor$,

i verify that if $k \in \mathbb{Z}, x \in \mathbb{R}$, then $f(x+k) = f(x) + k$ by taking

a $x = 4.25, k = 6$ b $x = -3.21, k = 7$ c $x = 8, k = -11$

ii verify that $f(x) + f(y) \leq f(x + y) \leq x + y$, using

a $x = 4.25, y = 6.32$ b $x = -2.01, y = \pi$ c $x = 4, y = -6.24$

iii verify that $f(x) \leq x < f(x) + 1$ by taking

a $x = 2.5$ b $x = -3.54$ c $x = 4$

3 Let $a = x - \lfloor x \rfloor$.

a Using Question 2iii above, show that $0 \leq a < 1$.

b Show that $x = \lfloor x \rfloor + a, \quad 0 \leq a < 1$.

c Show that $f(x + k) = f(x) + k$, when $k \in \mathbb{Z}$, $x \in \mathbb{R}$ using 3b.

1.3 CLASSIFICATION OF FUNCTIONS

1.3.1 One-to-One Functions

ACTIVITY 1.10

Which of the following is one-to-one?

$f = \{(a, 1), (b, 3), (c, 3), (d, 2)\}$; $g = \{(a, 4), (b, 2), (c, 3), (d, 1)\}$

Definition 1.8

A function f: A $\to$ B is said to be **one-to-one** (**an injection**), if and only if, each element of the range is paired with exactly one element of the domain, i.e.,

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, for any x_1 and $x_2 \in$ A.

Note:

This is the same as saying $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

Example 1 Show that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is one-to-one.

Solution Let $x_1, x_2 \in \mathbb{R}$ be any two elements such that $f(x_1) = f(x_2)$.

Then, $2x_1 = 2x_2 \Rightarrow \frac{1}{2}(2x_1) = \frac{1}{2}(2x_2) \Rightarrow x_1 = x_2$

Thus, f is one-to-one.

Example 2 Show that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not one-to-one.

Solution Take $x_1 = 2$ and $x_2 = -2$.

Obviously, $x_1 \neq x_2$ i.e $2 \neq -2$

But $f(x_1) = f(2) = 2^2 = 4 = (-2)^2 = f(-2) = f(x_2)$

This means there are numbers $x_1, x_2 \in \mathbb{R}$ for which $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ does not hold.

Thus, f is not one-to-one.

When the graph of $f: \mathbb{R} \to \mathbb{R}$ is given, i.e. f is a graphical function, there is another way of checking its one-to-oneness.

The horizontal line test:

A function $f: A \to B$ is one-to-one, if and only if any horizontal line crosses its graph at most once.

Example 3 Using the horizontal line test, show that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is one-to-one.

Solution From Figure 1.16, it is clear that any horizontal line crosses $y = 2x$ at most once. Hence, $f(x) = 2x$ is a one-to- one function.

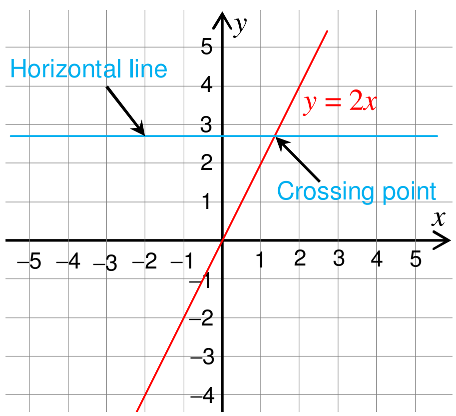


Figure 1.16 *Graph of* $f(x) = 2x$.

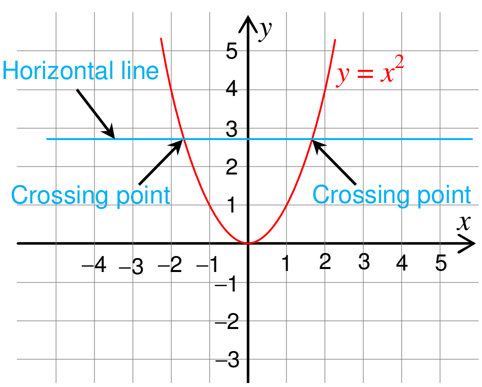


Figure 1.17 *Graph of* $f(x) = x^2$.

Example 4 Using the horizontal line test, show that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not one-to-one.

Solution A horizontal line crosses the graph of $y = x^2$ at two points (Figure 1.17). Thus, f is not one- to- one.

Example 5 Which of the following are one-to-one functions?

a $F = \{(x, y): y \text{ is the father of } x\}$

b $H = \{(x, y): y = |x - 2|\}$

c $G = \{(x, y): x \text{ is a dog and } y \text{ is its nose}\}$

Solution Only G is one-to-one.

Exercise 1.8

1 Which of the following functions are one-to-one?

a $f = \{(1, 5), (2, 6), (3, 7), (4, 8)\}$

b $f = \{(-2, 2), (-1, 3), (0, 1), (4, 1), (5, 6)\}$

c $f = \{(x, y): y \text{ is a brother of } x\}$

d $g = \{(x, y): x \text{ is a child and } y \text{ is his/her age}\}$

e $h: \mathbb{R} \to \mathbb{R}$, $h(x) = 3x - 2$.

f $h: (0, \infty) \to \mathbb{R}$, $h(x) = \log_5 x$.

g $f: \mathbb{R} \to \mathbb{R}$, given by $f(x) = |x - 1|$.

2 Let a, b, c, d be constants with $ad - bc \neq 0$, and $f(x) = \dfrac{ax + b}{cx + d}$. Check whether or not f is one-to-one.

1.3.2 Onto Functions

Definition 1.9

A function $f: A \to B$ is **onto** (**a surjection**), if and only if, Range of f = B.

Example 6 Let f be defined by the Venn diagram in Figure 1.18 below.

Range of f = B. Therefore, f is onto.

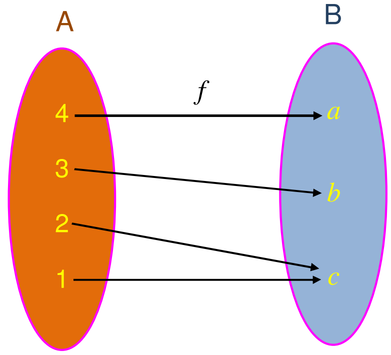


Figure 1.18

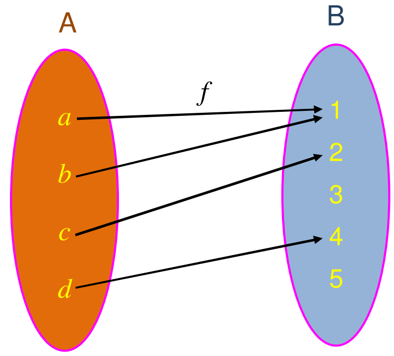


Figure 1.19

Example 7 In Figure 1.19 above,

Range of $f = \{1, 2, 4\} \Rightarrow$ Range of $f \neq$ B

Thus, f is not onto.

✍Note:

Let f: $A \rightarrow B$ be a function.

Range of f = B means for any $y \in$ B, there is $x \in$ A, such that $y = f(x)$.

So, to show f is onto, if possible, show that for any y, there is $x \in$ A such that $f(x) = y$.

To show f is not onto, find $y \in$ B that is not an image of any of the elements of A.

Example 8

a Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be $f(x) = x^2$

Take $y = -4$. Since for any $x \in \mathbb{R}$, $x^2 \geq 0$, $x^2 \neq -4$. Thus f is not onto.

b Let $f : \mathbb{R} \rightarrow [0, \infty)$ be given by $f(x) = x^2$.

Take $y \in [0, \infty)$. Since $y \geq 0$, for all $x \in \mathbb{R}$, $x^2 \in [0, \infty)$. Thus, $x^2 = y$ has a solution. Indeed, if $x = \sqrt{y}$, then, $f(x) = f\left(\sqrt{y}\right) = \left(\sqrt{y}\right)^2 = y$

Thus, f is onto.

Definition 1.10

A function f : A $\rightarrow$ B is a **one-to-one correspondence** (**a bijection**), if and only if, f is one-to-one and onto.

Example 9 Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 5x - 7$. Show that f is a one-to- one correspondence.

Solution Let $x_1, x_2 \in \mathbb{R}$, such that $f(x_1) = f(x_2)$

$$\Rightarrow 5x_1 - 7 = 5x_2 - 7 \Rightarrow 5x_1 - 7 + 7 = 5x_2 - 7 + 7$$

$$\Rightarrow 5x_1 = 5x_2 \Rightarrow x_1 = x_2$$

So, f is one-to-one.

Let $y \in \mathbb{R}$. Is there $x \in \mathbb{R}$ such that $y = f(x)$?

If there is, it can be found by solving $y = f(x) = 5x - 7$

$$\Rightarrow y + 7 = 5x \Rightarrow x = \frac{y+7}{5}.$$

So for any $y \in \mathbb{R}$, take $x = \frac{y+7}{5} \in \mathbb{R}$.

Then $f(x) = f\left(\frac{y+7}{5}\right) = 5\left(\frac{y+7}{5}\right) - 7 = y$

So f is onto.

Therefore, f is a one-to-one correspondence.

Example 10 Check if the following function is a one-to-one correspondence.

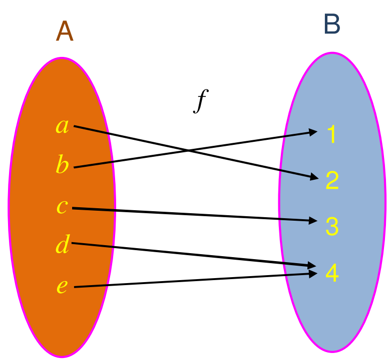


Figure 1.20

Solution f is onto, because range of $f = \{1, 2, 3, 4\} = B$.

But f is not one-to-one, because $f(d) = f(e) = 4$, while $d \neq e$.

So, f is not a one-to-one correspondence.

Example 11 Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 3^x$. Check whether or not f is a one-to-one correspondence.

Solution For any $x_1, x_2 \in \mathbb{R}$,

$$f(x_1) = f(x_2) \Rightarrow 3^{x_1} = 3^{x_2} \Rightarrow \frac{3^{x_1}}{3^{x_2}} = 1 \Rightarrow 3^{x_1 - x_2} = 1 = 3^0$$

$$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

Thus, f is one-to-one. But, f is not onto, because negative numbers cannot be images. For instance, take $y = -4$.

Since $3^x > 0$, for every $x \in \mathbb{R}$, it is not possible to have $x \in \mathbb{R}$, for which

$3^x = -4$.

Thus, f is not onto

Therefore, f is not a one-to-one correspondence.

Exercise 1.9

1 Which of the following functions are onto?

a $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x + 5$

b $g: [0, \infty) \to \mathbb{R}$, $g(x) = 3 - \sqrt{x}$

c

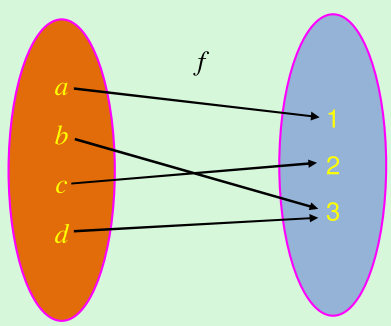


Figure 1.21

d $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$

e $h: \mathbb{R} \to \mathbb{R}$, $h(x) = |x - 1|$

2 For each of the following functions, give the set B, for which $f: \mathbb{R} \to$ B is onto.

a $f(x) = x^2 + 2$

b $f(x) = |x| + 5$

c $f(x) = 3|x|$

d $f(x) = 1 - 3|x|$

3 Show whether each of the following functions is a one-to-one correspondence or not.

a $f: \mathbb{R} \to \mathbb{R},\ f(x) = \dfrac{3x+1}{5}$

b $g: [0, \infty) \to [0, \infty),\ g(x) = \sqrt{x}$

c $h: \mathbb{R} \to (0, \infty),\ h(x) = 5^x$

d $f: [1, \infty) \to [0, \infty),\ f(x) = (x-1)^2 + 1$

4 Find a one-to-one correspondence between the following pairs of sets.

a $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$.

b $A = \{-1, -2, -3, \ldots, -50\}$, $B = \{1, 2, 3, \ldots, 50\}$.

c $A = \mathbb{N}$ and $B = \{5, 8, 11, \ldots\}$.

1.4 COMPOSITION OF FUNCTIONS

Combination of functions

Note:

Recall the following.

✓ Let f and g be two functions. Then,

$(f+g)(x) = f(x) + g(x)$; $(f-g)(x) = f(x) - g(x)$;

$(fg)(x) = f(x)\,g(x)$; $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$; where $g(x) \neq 0$.

✓ Domain of $(f+g)$ = Domain of $(f-g)$

= Domain of (fg) = Domain of $f \cap$ Domain of g.

✓ Domain of $\left(\dfrac{f}{g}\right)$ = (Domain of $f \cap$ Domain of g) \ $\{x: g(x) = 0\}$

Definition 1.11

Let $f: A \to B$ and $g: B \to C$ be functions. Then, the composition of g by f, denoted by gof, is given as $(gof)(x) = g(f(x))$.

Example 1 Given the Venn diagram in Figure 1.22, find

a $(gof)(a)$ **b** $(gof)(d)$

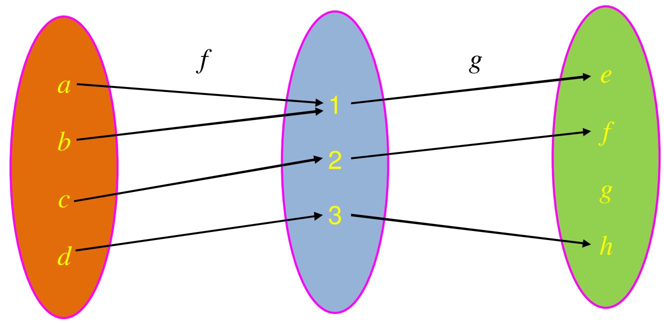


Figure 1.22

Solution $(gof)(a) = g(f(a)) = g(1) = e$ and $(gof)(d) = g(f(d)) = g(3) = h$

Example 2 Given the Venn diagram in Figure 1.23, find

a $(gof)(b)$ **b** $(gof)(c)$

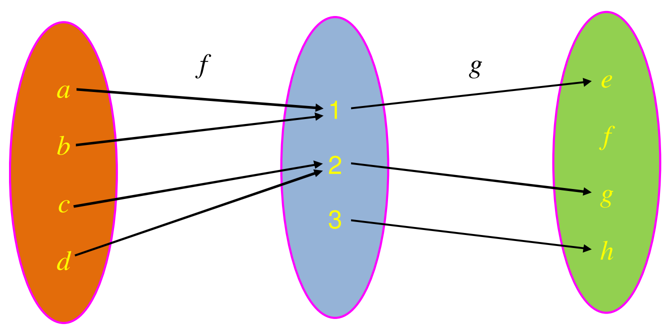


Figure 1.23

Solution $(gof)(b) = g(f(b)) = g(1) = e$ and $(gof)(c) = g(f(c)) = g(2) = g$

Example 3 Let $f(x) = 2x + 1$, $g(x) = x^3$. Give $(fog)(x)$ and $(gof)(x)$.

Solution $(fog)(x) = f(x^3) = 2x^3 + 1$, while $(gof)(x) = g(2x + 1) = (2x + 1)^3$

Example 4 Give $(fog)(x)$, $(gof)(x)$, $(fof)(x)$, $(gog)(x)$, if they exist, for

$f(x) = \log_5 x$, $g(x) = x^2 + 2$.

Solution Range of $f = \mathbb{R}$ Domain of $g = \mathbb{R}$.

$\Rightarrow (gof)(x)$ exists and $(gof)(x) = g(\log_5 x) = (\log_5 x)^2 + 2$.

Range of $g = [2, \infty)$. Domain of $f = (0, \infty)$ and hence

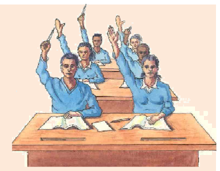
Domain of $f \cap$ Range of $g \neq \varnothing$.

$\Rightarrow fog$ exists and $(fog)(x) = \log_5(x^2 + 2)$

$(fof)(x) = \log_5 f(x) = \log_5(\log_5 x)$ can be defined only if $\log_5 x > 0$. i.e. if and only if $x > 1$.

$(gog)(x) = g(x^2 + 2) = (x^2 + 2)^2 + 2$. Here x can be any real number.

ACTIVITY 1.11



1 Let $f(x) = x + 1$ and $g(x) = \sqrt{x}$.

- **a** Give *fog* and *gof*.
- **b** Find the
  - **i** domain of *fog*
  - **ii** domain of *gof*
  - **iii** range of *fog*
  - **iv** range of *gof*

2 Let $f(x) = x^2 - 1$ and $g(x) = |x|$

- **a** Give *fog*, *gof*.
- **b** What is the domain of *gof* ?
- **c** Sketch the graph of *gof*.

3 Let $f(x) = \log_2 x$ and $g(x) = |x|$.

- **a** Give *fog* and *gof*.
- **b** Give the domain of *fog* and the domain of *gof*.
- **c** Sketch the graph of both the functions *fog* and *gof*.

Example 5 Let $f: \mathbb{R} \to [0, \infty)$ be given by $f(x) = 2^x$ and $g: [0, \infty) \to [0, \infty)$ be given by $g(x) = \sqrt{x}$. Then, find $(gof)(x)$ and the domain of $(gof)(x)$.

Solution $(gof)(x) = g(2^x) = \sqrt{2^x} = 2^{\frac{x}{2}}$. Domain of gof = Domain of $f = \mathbb{R}$.

Example 6 Let $f(x) = 5x + 4$ and $gof(x) = 7x - 1$. Find $g(x)$.

Solution Since *gof* and *f* are linear, try a linear function $g(x) = ax + b$.

$$g(f(x)) = g(5x + 4) = a(5x+4)+b = 5ax + 4a + b$$

Now, $g(f(x)) = 7x - 1 \Rightarrow 5ax + 4a + b = 7x - 1 \Rightarrow 5a = 7 \Rightarrow a = \frac{7}{5}$ and

$$4a + b = -1 \Rightarrow b = -1 - 4a \Rightarrow b = -1 - \frac{28}{5} = -\frac{33}{5}$$

Thus, $g(x) = \frac{7}{5}x - \frac{33}{5}$.

Exercise 1.10

1 Let $f(x) = 9x - 2$ and $g(x) = \sqrt{3x+7}$. Evaluate the following.

a $g(3) - g(-2)$ **b** $(g(-1))^2$ **c** $\dfrac{f(x) - f(0)}{x}$

2 Let $f(x) = 9x - 2$; $g(x) = \sqrt{3x+7}$. Find each of the following.

a $(f + g)(-2)$ **b** $\dfrac{f}{g}(7)$ **c** Domain of $(f + g)$

d Domain of fg **e** Domain of $\dfrac{f}{g}$

3 Find $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$ and $\left(\dfrac{f}{g}\right)(x)$ for the following.

a $f(x) = \dfrac{5x}{2x-1}$; $g(x) = \dfrac{-6x}{2x-1}$ **b** $f(x) = \sqrt{x+1}$; $g(x) = \dfrac{1}{\sqrt{x+1}}$

4 Let $f(x) = 3x - 2$; $g(x) = 5x + 1$. Compute the indicated values.

a $(fog)(3)$ **b** $(fof)(0)$ **c** $(gof)(-5)$

d $(gog)(-7)$ **e** $(fogof)(2)$

5 Find

i $(fog)(x)$ **ii** $(gof)(x)$

iii $(fof)(x)$ **iv** $(gog)(x)$, if they exist, for

a $f(x) = 2x - 1$; $g(x) = 4x + 2$ **b** $f(x) = x^2$; $g(x) = \sqrt{x}$

c $f(x) = 1 - 5x$; $g(x) = |2x + 3|$ **d** $f(x) = 3x$; $g(x) = 2^x$

6 Let $f(x) = 3x$, $g(x) = |x|$ and $h(x) = \sqrt{x}$. Express each function below as a composition of any two of the above functions.

a $l(x) = \sqrt{3x}$ **b** $k(x) = 3|x|$ **c** $t(x) = \sqrt{|x|}$

7 Express each function f as a composite of two simpler functions h and g. i.e. $f = hog$.

a $f(x) = \sqrt{3x+1}$ **b** $f(x) = 16x^2 - 3$ **c** $f(x) = 2^{3x^2+1}$

d $f(x) = 5 \times 2^{2x} + 3$ **e** $f(x) = x^4 - 6x^2 + 6$

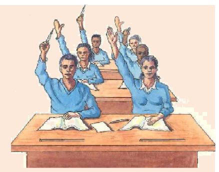
8 Let $f(x) = 4x + 1$ and $g(x) = 3x + k$, find the value of k for which $(fog)(x) = (gof)(x)$.

9 If $f(x) = ax + b$, $a \neq 0$, find g such that $(gof)(x) = x$.

10 Given $f(x) = x^4$ and $g(x) = 2x + 3$, show that $(fog)(x) \neq (gof)(x)$, in general.

1.5 INVERSE FUNCTIONS AND THEIR GRAPHS

ACTIVITY 1.12



Give the inverses of each of the following:

a $f = \{(x, y): y = 3x - 4\}$. Is f^{-1} a function?

b $R = \{(x, y): y \geq 3x - 4\}$. Is R^{-1} a function?

c $f = \{(x, y): y = x^2\}$. Is f^{-1} a function?

d $g = \{(x, y): y = \log_2 x\}$. Is g^{-1} a function?

From your investigation, you should have noticed that:

Note:

f^{-1} is a function, if and only if f is one-to-one.

Example 1 Is the inverse of $f(x) = x^3 - x + 1$ a function?

Solution Domain of $f = \mathbb{R}$ and for $1, -1 \in$ Domain of f,

$f(1) = 1 - 1 + 1 = 1 = f(-1)$. This implies f is not one-to-one.

Therefore f^{-1} is not a function.

Notation: If the inverse of f is g, then g is denoted by f^{-1}. In this case f is called **invertible**.

Steps to find the inverse of a function *f*

1. Interchange x and y in the formula of f.
2. Solve for y in terms of x.
3. Write $y = f^{-1}(x)$.

Example 2 Find the inverse of each of the following functions.

a $f(x) = 4x - 3$. **b** $f(x) = 1 - 3x$ **c** $f(x) = \dfrac{x}{x-1}, x \neq 1$.

Solution

a $f = \{(x, y): y = 4x - 3\}$ and

$$f^{-1} = \{(x, y): x = 4y - 3\} = \left\{(x, y): \frac{x+3}{4} = y\right\} \Rightarrow f^{-1}(x) = \frac{x+3}{4}$$

b $f = \{(x, y): y = 1 - 3x\}$

$$\Rightarrow f^{-1} = \{(x, y): x = 1 - 3y\} = \left\{(x, y): y = \frac{1-x}{3}\right\}.$$

Therefore $f^{-1}(x) = \dfrac{1-x}{3}$.

c $f = \{(x, y): y = \dfrac{x}{x-1}, x \neq 1\}$

$$f^{-1} = \{(x, y): x = \frac{y}{y-1}, x \neq 1\} = \{(x, y): x(y-1) = y, x \neq 1\}$$

$$= \{(x, y): y(x-1) = x, x \neq 1\}$$

$$= \{(x, y): y = \frac{x}{x-1}, x \neq 1\}$$

Definition 1.12

The function $I: A \to A$, given by $I(x) = x$ is called the **identity function**.

Note:

If $f: A \to A$, and $I: A \to A$, then $(I \circ f)(x) = I(f(x)) = f(x)$, for every $x \in A$.

Again, $(f \circ I)(x) = f(I(x)) = f(x)$, for every $x \in A$

We can define the inverse of a function using the composition of functions as follows.

Definition 1.13

A function g is said to be an **inverse** of a function f, if and only if,

$$g(f(x)) = I(x) \text{ and } f(g(x)) = I(x)$$

Example 3 Show whether or not each of the following pairs of functions are inverses of each other.

a $f: \mathbb{R} \to (0, \infty)$ given by $f(x) = 2^x$ and

$g: (0, \infty) \to \mathbb{R}$ given by $g(x) = \log_2 x$.

b $f(x) = \dfrac{x+1}{x+2}, x > -2$ and $g(x) = \dfrac{1-2x}{x-1}, x \neq 1$

c $f(x) = \dfrac{x+5}{x+1}; x \neq -1$ and $g(x) = \dfrac{5-x}{x+1}; x \neq -2$

Solution

a $(fog)(x) = 2^{\log_2 x} = x$ and $(gof)(x) = \log_2 2^x = x = I(x)$:

Thus f and g are inverses of each other. i.e. $g = f^{-1}$ or $f = g^{-1}$

b $f(g(x)) = f\left(\frac{1-2x}{x-1}\right) = x = \mathrm{I}(x)$ and $g(f(x)) = g\left(\frac{x+1}{x+2}\right) = x = \mathrm{I}(x)$.

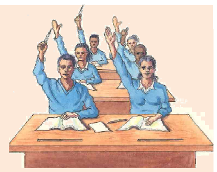
Thus f and g are inverses of each other, i.e. $g = f^{-1}$ or $f = g^{-1}$.

c $f(g(x)) = f\left(\frac{5-x}{x+2}\right) = \frac{4x+15}{7} \neq \mathrm{I}(x)$ and

$g(f(x)) = g\left(\frac{x+5}{x+1}\right) = \frac{4x}{3x+7} \neq \mathrm{I}(x)$

Hence f and g are not inverses of each other.

ACTIVITY 1.13



Recall that the graph of the inverse of a relation is obtained by reflecting the graph of the relation with respect to the line $y = x$.

For each of the following, sketch the graph of f and f^{-1} using the same coordinate axes.

a $f(x) = 2x + 3$ **b** $f(x) = x^3$

From Activity 1.14, you may have observed that the graph of f^{-1} can be obtained by reflecting the graph of f with respect to the line $y = x$.

Exercise 1.11

1 Determine the inverse of each of the following functions. Is the inverse a function?

a $f(x) = \log_3 2x$ **b** $h(x) = -5x + 13$

c $g(x) = 1 + \sqrt{x}$ **d** $k(x) = (x-2)^2$

2 Give the domain of each inverse in Question 1 above.

3 Are the following functions inverses of each other (*in their respective domain*)?

a $f(x) = 3x + 2$; $g(x) = \frac{x-2}{3}$ **b** $f(x) = x^3$; $g(x) = \sqrt[3]{x}$

c $f(x) = \sqrt{x}$; $g(x) = x^2$ **d** $f(x) = \sqrt[3]{x+8}$ and $g(x) = x^3 - 8$

4 Which of the following functions are invertible? If they are not, can you restrict the domain to make them invertible?

a $f(x) = x^3$ **b** $g(x) = 4 - x^2$

c $h(x) = -\frac{1}{3}x + 5$ **d** $f(x) = \log x^2$

5 Which of the following functions are invertible?

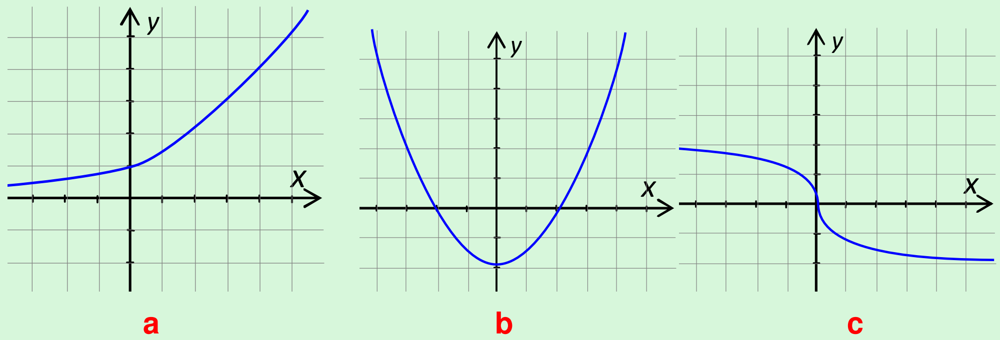


Figure 1.24

6 Sketch f^{-1} for each of the following functions.

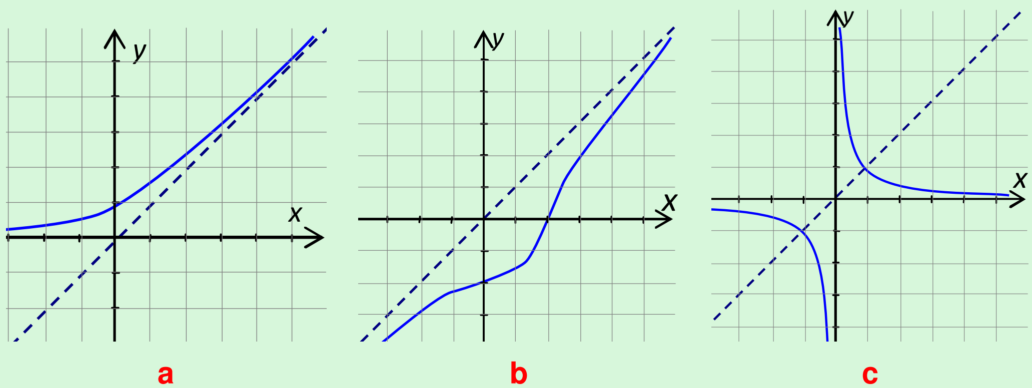


Figure 1.25

Key Terms

combination of functions	modulus (absolute value)
composite function	one-to-one correspondence
cusp	one-to-one function
domain	onto function
function	parity
greatest integer (floor) function	power function
horizontal line test	range
identity function	relation
inflection point	signum (sgn) function
inverse function	vertical line test

Summary

1 A **relation** from A to B is any subset of A × B.

2 $(f \pm g)(x) = f(x) \pm g(x)$; $(fg)(x) = f(x)g(x)$; $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ provided that $g(x) \neq 0$.

3 $R^{-1} = \{(b, a): (a, b) \in R\}$

4 Domain of R^{-1} = Range of R and Range of R^{-1} = Domain of R.

5 A **function** is a relation in which no two of the ordered pairs in it have the same first element.

6 $f(x) = ax^r$, $r \in \mathbb{Q}$ is called a **power function**.

7 $f(x) = ax^{\frac{m}{n}}$, m even and n odd has a **cusp** at the origin.

8 $|x| = \begin{cases} x, \text{ for } x \geq 0 \\ -x, \text{ for } x < 0 \end{cases}$

9 $|x| = \sqrt{x^2}$

10 $\operatorname{sgn} x = \begin{cases} 1, \text{ for } x > 0 \\ 0, \text{ for } x = 0 \\ -1, \text{ for } x < 0 \end{cases}$

11 The **floor function** or the greatest integer function $f(x) = \lfloor x \rfloor$ maps $\mathbb{R}$ into $\mathbb{Z}$.

12 f is **one-to-one**, if and only if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, for any $x_1, x_2 \in$ Domain of f.

13 A numerical function f is one-to-one, if and only if no horizontal line crosses the graph of f more than once.

14 f : A→B is **onto**, if and only if Range of f = B.

15 f : A→B is a **one-to-one correspondence**, if and only if f is one-to-one and onto.

16 $(fog)(x) = f(g(x))$

17 Domain of $(fog) \subseteq$ Domain of g.

18 f^{-1} is a function, if f is one-to-one.

19 g and f are inverse functions of each other, if and only if $g(f(x)) = x$ and $f(g(x)) = x$.

20 To find f^{-1}

- ✓ Write $y = f(x)$.
- ✓ Interchange x and y in the above equation to obtain $x = f(y)$.
- ✓ Solve for y and write $y = f^{-1}(x)$.

Review Exercises on Unit 1

1 Find the inverse of each relation and determine whether the inverse is a function.

a R ={(2, –2), (–3, 3), (–4, –4)} **b** R = {(2, 1), (2, 3), (2, 7)}

2 Find the inverse of each function. Is the inverse a function?

a $f(x) = 2x + 3$ **b** $f(x) = x^2 - 9$

c $f(x) = (x^2 - 9)^2$ **d** $f(x) = \frac{\sqrt{x}}{3}$

3 Find the domain of $f(x) = \sqrt{|x| - x}$.

4 **a** Give the intersection points of $y = x^5$ and $y = x^7$.

b Are these points common to $y = x^n$, where n is an odd natural number?

c For each of the following functions, what is the effect of 4, when $f(x)$ is compared with $y = x^3$?

i $f(x) = 4x^3$ **ii** $f(x) = x^3 + 4$

d For $f(x) = x^3$, compare $f(a{\cdot}b)$ and $f(a){\cdot}f(b)$ for any $a, b \in \mathbb{R}$. What do you notice?

e Is the property $f(a{\cdot}b) = f(a){\cdot}f(b)$ for any $a, b \in \mathbb{R}$ generally true for any $f(x) = x^n, n \in \mathbb{R}$?

5 Draw the graph of $y = |f(x)|$ using the graph of $y = f(x)$, for each of the following:

a $f(x) = x + 1$ **b** $f(x) = \log x$ **c** $f(x) = x^3$

6 **a** Show that $y = \text{sgn}\, x$ is odd.

b If $h(x) = \frac{1}{2}(\text{sgn}\, x + 1)$, show that $h(-x) + h(x) = 1$

c Express $h(x)$ in terms of x, by taking $x > 0$, $x = 0$ and $x < 0$.

7 For $f(x) = \lfloor x \rfloor$, verify that $f(x + y) \leq f(x) + f(y) + 1$, by taking

a $x = -3.9$; $y = -16.4$ **b** $x = 3.9$; $y = -16.4$

c $x = -3.9$; $y = 16.4$ **d** $x = 3.9$; $y = 16.4$

8 Check if $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$, by taking different values of x.

9 Find fog, fof, gof, gog for

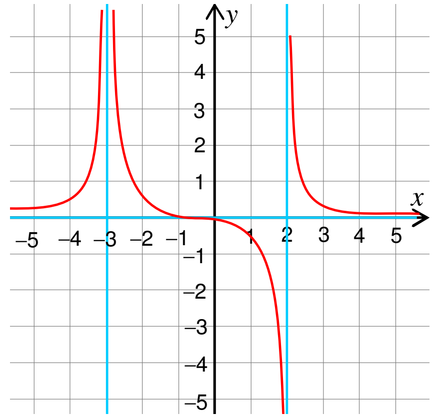
a $f(x) = 1 + 2x$; $g(x) = |x|$ **b** $f(x) = \log x$; $g(x) = 3x + 1$

10 What is the domain of each composite function in Question 9?

11 Determine whether or not each pair of functions are inverses of each other.

a $f(x) = 2x - 4$; $g(x) = \frac{x+4}{2}$ **b** $f(x) = 2x + 5$; $g(x) = \frac{3x-5}{2}$

Unit 2



RATIONAL EXPRESSIONS AND RATIONAL FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *know methods and procedures in simplifying rational expressions.*
- *understand and develop efficient methods in solving rational equations and inequalities.*
- *know basic concepts and specific facts about rational functions.*

Main Contents

INTRODUCTION

We now turn our attention to fractional forms. A quotient of two algebraic expressions, division by 0 excluded, is called a **fractional expression**. If both the numerator and denominator are polynomials, the fractional expression is called a **rational expression**.

Just as rational numbers are defined in terms of quotients of integers, rational expressions are defined in terms of quotients of polynomials.

HISTORICAL NOTE

John Bernoulli (1667 – 1748)

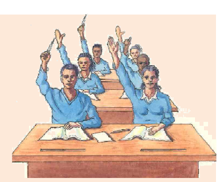
The method of partial fractions was introduced by John Bernoulli, a Swiss Mathematician who was instrumental in the early developments of calculus. John Bernoulli was a professor at the University of Basel and taught many outstanding students, the most famous of whom was Leonhard Euler.

OPENING PROBLEM

Aberra, working alone, can paint a small house in 6 hours. Genet can paint the same house in 12 hours. If they work together, how long will it take them to paint the house?

2.1 SIMPLIFICATION OF RATIONAL EXPRESSIONS

ACTIVITY 2.1



Determine the domain (or universal set) of each of the following expressions.

a $x^2 + 3x - 4$ **b** $\log(x + 2)$ **c** $\sqrt{1-5x}$ **d** $\dfrac{3x}{x+5}$

2.1.1 Rational Expressions

Definition 2.1

A **rational expression** is the quotient $\frac{P(x)}{Q(x)}$ of two polynomials $P(x)$ and $Q(x)$, where $Q(x) \neq 0$. $P(x)$ is called the **numerator** and $Q(x)$ is called the **denominator**.

Some examples of rational expressions are the following (recall, a non-zero constant is a polynomial of degree 0):

Example 1 Which of the following are rational expressions?

a $\frac{x-2}{2x^2-3x+4}$ **b** $\frac{1}{x^4-1}$ **c** $\frac{x^3+3x-6}{4}$ **d** $\sqrt{1-5x}$

Solution All except d are rational expressions

Example 2 Evaluate the rational expression $\frac{2x-5}{3x+9}$ for the given values of x:

a $x=5$ **b** $x=-6$

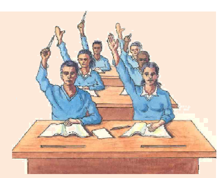
Solution

a At $x=5$, $\frac{2x-5}{3x+9} = \frac{2(5)-5}{3(5)+9} = \frac{10-5}{15+9} = \frac{5}{24}$

b At $x=-6$, $\frac{2x-5}{3x+9} = \frac{2(-6)-5}{3(-6)+9} = \frac{-12-5}{-18+9} = \frac{-17}{-9} = \frac{17}{9}$

Domain of a rational expression

ACTIVITY 2.2



Do the following activities:

a Find the domain of $\frac{x^2+2x}{5x}$.

b Factorize the numerator and denominator of $\frac{x^2+2x}{5x}$

c Simplify $\frac{x^2+2x}{5x}$.

d When are $\frac{x^2+2x}{5x}$ and its simplified form equal?

e Do steps a – d for the rational expression $\frac{9x^2-4}{9x^2+9x-10}$.

✍**Note:**

In Example 2 above, since the denominator $3x + 9 = 0$, for $x = -3$, $\frac{2x-5}{3x+9}$ is undefined when $x = -3$. Therefore, the domain of $\frac{2x-5}{3x+9}$ is $\{x : x$ is a real number and $x \neq -3\}$.

Steps to find the domain of a rational expression:

1 Set the denominator of the expression equal to zero and solve.

2 The domain is the set of all real numbers except those values found in step 1.

Example 3 Find the domain of each of the following rational expressions:

a $\frac{19}{3x}$ **b** $\frac{x^2-9}{x^2-7x+10}$

Solution

a Set the denominator equal to zero and solve: $3x = 0 \Rightarrow x = 0$.

Thus, the domain is $\{x : x$ is a real number and $x \neq 0\}$ or $\mathbb{R} \setminus \{0\}$.

b Set the denominator equal to zero and solve:

$$x^2 - 7x + 10 = 0 \quad \text{(factor)}$$
$$(x-5)(x-2) = 0 \quad \text{(set each factor equal to 0 and solve)}$$
$$x - 5 = 0 \text{ or } x - 2 = 0$$
$$x = 5 \text{ or } x = 2$$

Thus, the domain is $\{x : x$ is a real number and $x \neq 2, x \neq 5\} = \mathbb{R} \setminus \{2, 5\}$

Fundamental Property of Fractions

If a, b and k are real numbers with b, $k \neq 0$, then $\frac{ka}{kb} = \frac{a}{b}$.

✍**Note:**

Using the above property and eliminating all common factors from the numerator and denominator of a given fraction, is referred to as reducing (or simplifying) the fraction to its lowest term.

Definition 2.2

We say that a rational expression is **reduced to lowest terms** (or **in its lowest terms** or **in simplest form**), if the numerator and denominator do not have any common factor other than 1.

✍Note:

It is important to emphasize that $\frac{9x^2-4}{9x^2+9x-10}=\frac{3x+2}{3x+5}$ only if $x\neq\frac{-5}{3}$ and $x\neq\frac{2}{3}$. Though $\frac{3x+2}{3x+5}$ is undefined at $x=\frac{-5}{3}$ only, the original expression $\frac{9x^2-4}{9x^2+9x-10}$ is undefined at $x=\frac{-5}{3}$ and $x=\frac{2}{3}$. We are only allowed to reduce $\frac{3x-2}{3x-2}$ to 1, provided that $3x-2\neq 0$.

To simplify a rational expression:

1 Find the domain.

2 Factorize the numerator and denominator completely.

3 Divide the numerator and denominator by any common factor (i.e. cancel like terms).

Example 4 Simplify the following.

a $\frac{2y^2+6y+4}{4y^2-12y-16}$. **b** $\frac{x^4+18x^2+81}{x^2+9}$. **c** $\frac{1-a}{7a^2-7}$.

Solution

a The universal set is $\mathbb{R}\setminus\{-1, 4\}$.

Thus, $\frac{2y^2+6y+4}{4y^2-12y-16}=\frac{2(y+2)(y+1)}{4(y-4)(y+1)}=\frac{y+2}{2(y-4)}$, for $y\neq 4$ and $y\neq -1$.

b $\frac{x^4+18x^2+81}{x^2+9}=\frac{(x^2+9)(x^2+9)}{x^2+9}=x^2+9$, for all $x\in\mathbb{R}$.

c $\frac{1-a}{7a^2-7}=\frac{-a+1}{7(a^2-1)}=\frac{-(a-1)}{7(a-1)(a+1)}=-\frac{1}{7(a+1)}$, for $a\in\mathbb{R}\setminus\{-1, 1\}$.

Exercise 2.1

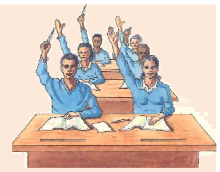
State the domain and simplify each of the following:

a $\frac{4x-12}{4x}$ **b** $\frac{6x^2+23x+20}{2x^2+5x-12}$ **c** $\frac{x^3+3x^2}{x+3}$

d $\frac{x^3-27}{x^4+3x^3-27x-81}$ **e** $\frac{x^2-5x+6}{3x^3-2x^2-8x}$ **f** $\frac{x^4-8x}{3x^3-2x^2-8x}$

2.1.2 Operations with Rational Expressions

ACTIVITY 2.3



Do the following in groups.

Perform the following operations on rational numbers.

a $\frac{5}{8}+\frac{7}{8}$ **b** $\frac{3}{4}+\frac{5}{6}$ **c** $\frac{11}{12}-\frac{4}{12}$ **d** $\frac{7}{10}-\frac{2}{5}$

✍Note:

Rational expressions obey the same rules as rational numbers, for addition, subtraction, multiplication and division.

Addition and subtraction of rational expressions

Let $P(x), Q(x)$, and $R(x)$ be polynomials such that $Q(x)\neq 0$, then

$$\frac{P(x)}{Q(x)}+\frac{R(x)}{Q(x)}=\frac{P(x)+R(x)}{Q(x)} \text{ and } \frac{P(x)}{Q(x)}-\frac{R(x)}{Q(x)}=\frac{P(x)-R(x)}{Q(x)}.$$

Example 5 For each rational expression, state the universal set and simplify.

a $\frac{x-5}{x+1}+\frac{x+5}{x+1}$ **b** $\frac{2x-3}{x^2+6x+9}-\frac{-x+6}{x^2+6x+9}$

c $\frac{3a+13}{a+4}-\frac{2a+7}{a+4}$ **d** $\frac{a^2-1}{a^2-7a+12}-\frac{8}{a^2-7a+12}$

Solution

a $\frac{x-5}{x+1}+\frac{x+5}{x+1}=\frac{(x-5)+(x+5)}{x+1}=\frac{2x}{x+1}$, for $x\neq -1$.

b $\frac{2x-3}{x^2+6x+9}-\frac{-x+6}{x^2+6x+9}=\frac{(2x-3)-(-x+6)}{x^2+6x+9}=\frac{3x-9}{x^2+6x+9}$, for $x\neq -3$.

c $\frac{3a+13}{a+4}-\frac{2a+7}{a+4}=\frac{(3a+13)-(2a+7)}{a+4}=\frac{a+6}{a+4}$, for $a\neq -4$.

d $\frac{a^2-1}{a^2-7a+12}-\frac{8}{a^2-7a+12}=\frac{(a^2-1)-8}{a^2-7a+12}=\frac{a^2-9}{a^2-7a+12}$

$$=\frac{(a-3)(a+3)}{(a-3)(a-4)}=\frac{a+3}{a-4}, \text{ for } a\neq 3 \text{ and } 4.$$

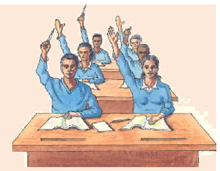
Exercise 2.2

Perform the indicated operations and simplify.

a $\dfrac{2x-3}{x^2+5x}-\dfrac{3x-5}{x^2+5x}$ **b** $\dfrac{x^2+3x}{x^2+2x-15}-\dfrac{2x+12}{x^2+2x-15}$

c $\dfrac{12}{5x}+\dfrac{x-2}{5x}$ **d** $\dfrac{6y+11}{4y^2+12y-7}-\dfrac{4y+4}{4y^2+12y-7}$

ACTIVITY 2.4



Do the following activities:

a Factorize the denominators and find the domain of $\dfrac{x-4}{x^2-9}+\dfrac{x+2}{x^2+11x+24}$.

b What is the least common multiple of the denominators?

c Apply the rule for addition of rational numbers to express

$\dfrac{x-4}{x^2-9}+\dfrac{x+2}{x^2+11x+24}$ in the form $\dfrac{P(x)}{Q(x)}$

d Simplify your result.

Steps to add and subtract rational expressions with unlike denominators:

1 Factorize the denominators completely.

2 Find the LCM.

3 Build each rational expression into an equivalent expression with the denominator equal to the LCM.

4 Add and subtract the numerators and write the result over the common denominator.

5 Simplify the numerator and factorize it to see if you can reduce it.

Example 6 Perform the indicated operations and simplify.

a $3+\dfrac{2}{3y+6}+\dfrac{y-3}{y^2-4}$ **b** $\dfrac{3}{2c-1}-\dfrac{1}{c+2}-\dfrac{5}{2c^2+3c-2}$

Solution

a We first find the LCM by factorizing each denominator:

$$\frac{3}{1}\cdot\frac{3(y-2)(y+2)}{3(y-2)(y+2)}+\frac{2(y-2)}{3(y+2)(y-2)}+\frac{3(y-3)}{3(y-2)(y+2)}$$

$$= \frac{9(y-2)(y+2)+2(y-2)+3(y-3)}{3(y-2)(y+2)}$$

$$= \frac{9y^2-36+2y-4+3y-9}{3(y-2)(y+2)} = \frac{9y^2+5y-49}{3y^2-12}\text{ , for y} \neq \text{ -2 and 2}$$

b Notice that, $2c^2 + 3c - 2 = (2c - 1)(c + 2)$. Thus, the LCM is $(2c - 1)(c + 2)$ and

$$\frac{3}{2c-1} - \frac{1}{c+2} - \frac{5}{2c^2+3c-2} = \frac{3}{2c-1} - \frac{1}{c+2} - \frac{5}{(2c-1)(c+2)}$$

$$= \frac{3(c+2)}{(2c-1)(c+2)} - \frac{2c-1}{(2c-1)(c+2)} - \frac{5}{(2c-1)(c+2)}$$

$$= \frac{3(c+2)-(2c-1)-5}{(2c-1)(c+2)} = \frac{c+2}{(2c-1)(c+2)} = \frac{1}{2c-1}\text{ for } c \neq -2 \text{ and } \frac{1}{2}.$$

Exercise 2.3

Perform the indicated operations and simplify.

a $\frac{25y^2}{5y-4} + \frac{16}{4-5y}$ **b** $\frac{1}{x^2} + \frac{1}{x^2+x}$ **c** $u+1+\frac{1}{u+1}$

d $\frac{2}{a^2} - \frac{3}{ab} + \frac{4}{b^2}$ **e** $\frac{3}{x^2-x} - \frac{2}{x^2+x-2}$ **f** $\frac{x}{(x+1)^2} + \frac{2}{x+1}$

g $\frac{-50x^2-55x+8}{15x^2+x-2} - \frac{25x}{5x+2} + \frac{25x^2+15x}{3x-1}$ **h** $\frac{6}{z+4} - \frac{2}{3z+12}$

ACTIVITY 2.5

Do the following in groups.

1 Perform the following operations on rational numbers.

a $\frac{3}{7}\times\frac{3}{5}$ **b** $\frac{5}{11}\times\frac{22}{45}$ **c** $\frac{4}{6}\div\frac{9}{12}$ **d** $\frac{7}{4}\div\frac{35}{6}$

2 What are the rules used to simplify the expressions?

Multiplication of rational expressions

If $P(x), Q(x), R(x)$ and $S(x)$ are polynomials such that $Q(x) \neq 0$, $S(x) \neq 0$, then

$$\frac{P(x)}{Q(x)} \cdot \frac{R(x)}{S(x)} = \frac{P(x)R(x)}{Q(x)S(x)}$$

Steps to multiply rational expressions:

1. Factorize the numerators and denominators completely.
2. Divide out all the common factors.
3. Multiply numerator with numerator and denominator with denominator to get the answer.

Example 7 Evaluate and simplify:

a $\frac{5x+5}{x-2}\cdot\frac{x^2-4x+4}{x^2-1}$ **b** $\frac{4x+20}{x^2+10x+25}\cdot\frac{x+2}{4x+8}$

Solution

a By first factorizing the numerators and denominators, we get:

$$\frac{5x+5}{x-2}\cdot\frac{x^2-4x+4}{x^2-1}=\frac{5(x+1)}{x-2}\cdot\frac{(x-2)(x-2)}{(x-1)(x+1)}=\frac{5(x-2)}{(x-1)},\text{for } x\neq -1,\ 1 \text{ and } 2.$$

b Factorizing the numerator and denominator yields:

$$\frac{4x+20}{x^2+10x+25}\cdot\frac{x+2}{4x+8}=\frac{4(x+5)}{(x+5)(x+5)}\cdot\frac{x+2}{4(x+2)}=\frac{1}{x+5},\ \text{for } x\neq -5 \text{ and } -2.$$

Division of rational expressions

If $P(x), Q(x), R(x)$ and $S(x)$ are polynomials such that $R(x)\neq 0$, $Q(x)\neq 0$, $S(x)\neq 0$ then

$$\frac{P(x)}{Q(x)}\div\frac{R(x)}{S(x)}=\frac{P(x)}{Q(x)}\cdot\frac{S(x)}{R(x)}=\frac{P(x)S(x)}{Q(x)R(x)}$$

Example 8 Perform the following operations and simplify:

a $\frac{36x^2-48x+16}{3x^2+13x-10}\div\frac{4x^2-12x+9}{2x^2+7x-15}$ **b** $\frac{a}{a-b}\div\frac{b}{a-b}$

Solution

a First, you have to invert the second fraction and multiply. Then factorize each expression and simplify.

$$\frac{36x^2-48x+16}{3x^2+13x-10}\times\frac{2x^2+7x-15}{4x^2-12x+9}=\frac{4(3x-2)(3x-2)}{(3x-2)(x+5)}\times\frac{(2x-3)(x+5)}{(2x-3)(2x-3)}$$

$$=\frac{4(3x-2)}{2x-3}\ \text{for } x\neq -5,\ \frac{2}{3} \text{ and } \frac{3}{2}.$$

b First, invert the second fraction and multiply:

$$\frac{a}{a-b}\div\frac{b}{a-b}=\frac{a}{a-b}\times\frac{a-b}{b}=\frac{a}{b},\ \text{for } a\neq b \text{ and } b\neq 0.$$

Exercise 2.4

1 Evaluate and simplify each of the following expressions. State their domains.

a $\frac{x^2-x-12}{x^2-9}\times\frac{3+x}{4-x}$ **b** $\frac{x^3-27}{x^2-9}\times\frac{x+3}{x^2+3x+9}$

2 Perform the indicated operations and simplify:

a $\frac{x^2-7x+12}{4-x}\times\frac{5}{x^2-9}$ **b** $\frac{2x^2-3x-2}{x^2-1}\div\frac{2x^2+5x+2}{x^2+x-2}$

c $\frac{x^2-x-6}{3x^2-12}\div\frac{x^2-3x}{2-x}$

2.1.3 Decomposition of Rational Expressions into Partial Fractions

So far, you have been combining rational expressions using addition, multiplication, subtraction and division rules. Next, you will consider the reverse process- decomposing a rational expression into simpler ones.

We obtain the sum of fractions $\frac{2}{x-2}$ and $\frac{3}{x+1}$ as follows:

$$\frac{2}{x-2}+\frac{3}{x+1}=\frac{5x-4}{(x-2)(x+1)}=\frac{5x-4}{x^2-x-2}$$

The reverse process of writing $\frac{5x-4}{x^2-x-2}$ as a sum or difference of simple fractions

(fractions with numerators of lesser degree than their denominators) is frequently important in calculus. Each such simple fraction is called a **partial fraction**, and the process itself is called **decomposition into partial fractions**.

Definition 2.3

In a rational expression $\frac{P(x)}{Q(x)}$, if the degree of $P(x)$ is less than that of $Q(x)$, then $\frac{P(x)}{Q(x)}$ is called a **proper rational expression**. Otherwise it is called **improper**.

From your previous knowledge of algebra, you know that any rational expression can be written as the sum of a polynomial and a proper rational expression.

To decompose a rational expression $\frac{P(x)}{Q(x)}$, the degree of $P(x)$ must be less than the degree of $Q(x)$. In a case where the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, you have only to divide $P(x)$ by $Q(x)$ to obtain $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$, where the degree of $R(x)$ is less than that of $Q(x)$. The decomposition is then done on $\frac{R(x)}{Q(x)}$.

Example 9 Express $\frac{2x^3 + 10x^2 - 3x + 1}{x + 3}$ as a sum of a polynomial and a proper rational fraction.

Solution Using long division,

$$2x^3 + 10x^2 - 3x + 1 = (x + 3)(2x^2 + 4x - 15) + 46.$$

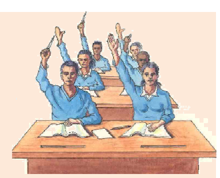
Thus, $\frac{2x^3 + 10x^2 - 3x + 1}{x + 3} = (2x^2 + 4x - 15) + \frac{46}{x + 3}$.

Moreover, you need to rely on the following definition to do the partial fraction decomposition:

Definition 2.4

Two polynomials of equal degree are equal to each other, if and only if the coefficients of terms of like degree are equal.

ACTIVITY 2.6



1 Factorize $x^3 - 3x^2 + 2x$

2 Factorize each of the following (if factorizable)

a $x^2 - 6x + 9$ **b** $15x^2 + 14x - 8$ **c** $x^2 - x + 2$

3 For each of the quadratic polynomials $ax^2 + bx + c$ in Question 2 above, find $b^2 - 4ac$. Which quadratic polynomial cannot be factorized further? Can we use the sign of $b^2 - 4ac$ to decide which quadratic polynomials can be factorized?

4 Factorize $x^4 + 7x^3 + 12x^2 - 7x - 13$.

✍Note:

$ax^2 + bx + c$ is not reducible in real numbers, if $b^2 - 4ac < 0$.

Theorem 2.1 Linear and quadratic factor theorem

For a polynomial with real coefficients, there always exists a complete factorization involving only linear and/or quadratic factors (raised to some power of natural number $k \geq 1$), with real coefficients, where the linear and quadratic factors are not reducible relative to real numbers.

So, once you have decided that partial fraction decomposition is to be done for a rational expression, you factorize the denominator as completely as possible. Then, for each factor in the denominator, you can use the following table to determine the term(s) you pick up in the partial fraction decomposition. The table gives the various cases that can arise.

	Factor in the Denominator	Corresponding term in the Partial Fraction
1	$ax+b$	$\frac{A}{ax+b}$, A constant
2	$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$, $A_1, A_2, \cdots, A_k$ are constants
3	ax^2+bx+c (with $b^2-4ac<0$)	$\frac{Ax+B}{ax^2+bx+c}$, A, B are constants
4	$(ax^2+bx+c)^k$ (with $b^2-4ac<0$)	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$, $A_1, A_2, \cdots, A_k$, $B_1, B_2, \cdots, B_k$ are constants.

Example 10 Decompose each of the following rational expressions into partial fractions:

a $\frac{5x+7}{x^2+2x-3}$ **b** $\frac{6x^2-14x-27}{(x+2)(x-3)^2}$ **c** $\frac{5x^2-8x+5}{(x-2)(x^2-x+1)}$

d $\frac{x^3-4x^2+9x-5}{(x^2-2x+3)^2}$ **e** $\frac{x^3}{(x+1)(x+2)}$

Solution

a The denominator $x^2+2x-3=(x-1)(x+3)$. The two factors $(x-1)$ and $(x+3)$ are distinct. Thus, we apply part 1 of the table to get:

$$\frac{5x+7}{x^2+2x-3}=\frac{5x+7}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}$$

To find the constants A and B, we combine the fractions on the right side of the above equation to obtain

$$\frac{5x+7}{(x-1)(x+3)} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

Since these expressions have the same denominator, their numerators must be equal.

Thus, $5x+7 = A(x+3)+B(x-1) = (A+B)x+3A-B$. Using Definition 2.4, we have

$$A+B=5 \text{ and } 3A-B=7, \text{ which gives } A=3 \text{ and } B=2.$$

Hence, $\dfrac{5x+7}{x^2+2x-3} = \dfrac{3}{x-1} + \dfrac{2}{x+3}$.

(This can easily be checked by adding the two fractions on the right.)

b Using parts 1 and 2 of the table, we write

$$\frac{6x^2-14x-27}{(x+2)(x-3)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

$$= \frac{A(x-3)^2 + B(x+2)(x-3) + C(x+2)}{(x+2)(x-3)^2}$$

Thus, $6x^2-14x-27 = A(x-3)^2 + B(x+2)(x-3) + C(x+2)$ which holds for all x.

In particular, if $x=3$, then $-15=5C$, which gives $C=-3$ and if $x=-2$, then $25=25A$, which gives $A=1$.

There are no other values of x that will cause terms on the right to be equal to zero. Since any value of x can be substituted to produce an equation relating A, B, and C, we let $x=0$ and obtain

$-27 = 9A-6B+2C$ (Substitute $A=1$ and $C=-3$)

$-27 = 9-6B-6 \Rightarrow B=5$

Thus, $\dfrac{6x^2-14x-27}{(x+2)(x-3)^2} = \dfrac{1}{x+2} + \dfrac{5}{x-3} - \dfrac{3}{(x-3)^2}$.

c For $x^2 - x + 1$, $b^2 - 4ac = -3 < 0$. Thus, it cannot be factorized further in the real numbers. Using, parts 1 and 3 of the table:

$$\frac{5x^2-8x+5}{(x-2)(x^2-x+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-x+1} = \frac{A(x^2-x+1)+(Bx+C)(x-2)}{(x-2)(x^2-x+1)}$$

Thus, for all x, $5x^2-8x+5 = A(x^2-x+1)+(Bx+C)(x-2)$.

If $x=2$, then 9 =3A which gives A = 3.

If $x=0$, then using $A=3$, we have 5 = 3 – 2C so that C = −1.

If $x = 1$, then using $A = 3$ and $C = -1$, you have B = 2.

Hence, $\dfrac{5x^2 - 8x + 5}{(x-2)(x^2 - x + 1)} = \dfrac{3}{x-2} + \dfrac{2x-1}{x^2 - x + 1}$.

d Since $x^2 - 2x + 3$ cannot be factorized further in the real numbers, you proceed to use part 4 of the table, as shown below

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2} = \frac{Ax + B}{x^2 - 2x + 3} + \frac{Cx + D}{(x^2 - 2x + 3)^2}$$

$$= \frac{(Ax + B)(x^2 - 2x + 3) + (Cx + D)}{(x^2 - 2x + 3)^2}$$

Thus, for all x, $x^3 - 4x^2 + 9x - 5 = (Ax + B)(x^2 - 2x + 3) + Cx + D$

$$= Ax^3 + (B - 2A)x^2 + (3A - 2B + C)x + (3B + D)$$

Equating coefficients of terms of like degree, we obtain

$A = 1$; $B - 2A = -4$; $3A - 2B + C = 9$ and $3B + D = -5$

From these equations we find that $A = 1$, $B = -2$, $C = 2$ and $D = 1$. Now you can write

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2} = \frac{x-2}{x^2 - 2x + 3} + \frac{2x+1}{(x^2 - 2x + 3)^2}.$$

e This is not a proper rational expression.

Note that $(x+1)(x+2) = x^2 + 3x + 2$. Divide x^3 by $x^2 + 3x + 2$. It gives a quotient $x - 3$ and remainder $7x + 6$.

Therefore, $\dfrac{x^3}{(x+1)(x+2)} = x - 3 + \dfrac{7x+6}{(x+1)(x+2)}$.

Now using the usual method, $\dfrac{7x+6}{(x+1)(x+2)} = \dfrac{-1}{x+1} + \dfrac{8}{x+2}$.

Hence, $\dfrac{x^3}{(x+1)(x+2)} = (x-3) - \dfrac{1}{x+1} + \dfrac{8}{x+2}$

Exercise 2.5

Write each of the following rational expressions in partial fractions:

a $\dfrac{7x+6}{x^2 + x - 6}$ **b** $\dfrac{5x+7}{(x-1)(x^2 + x + 2)}$ **c** $\dfrac{3x+5}{(x-2)^2}$

d $\dfrac{(x+3)^2}{(x^2+1)(x+3)}$ **e** $\dfrac{x^2 + 4x - 3}{x-2}$ **f** $\dfrac{7x^2 - 11x + 6}{(x-1)(2x^2 - 3x + 2)}$

2.2 RATIONAL EQUATIONS

You already know how to solve linear and quadratic equations. In this subunit, you will discuss the solution of rational equations.

ACTIVITY 2.7

State the universal set and solve each of the following equations.

a $\frac{2}{3} = \frac{x}{3}$ **b** $x + 2 - 3\,(x - 2) = 0$

c $\frac{x}{3} + \frac{3x}{4} = 2$ **d** $2\,(10x + 3) = 5x + 6$

e The staff members of a school agreed to contribute 200 Birr each to make up a fund to help needy students in the school. Since then, two new members have joined the staff, and as a result, each member's share has been reduced by 5 Birr. How many members are now on the staff?

Definition 2.5

A **rational equation** is an equation that can be reduced to the form $\frac{P(x)}{Q(x)} = 0$ where $P\,(x)$ and $Q\,(x)$ are polynomials and $Q\,(x) \neq 0$.

Note:

To solve a rational equation, you can multiply both sides of the equation by the LCM of the denominators for those values of the variable for which the LCM is non-zero. An important thing to keep in mind is that those values that cause the denominator to become 0 cannot be solutions to the equation. A number that looks to be a solution but causes the denominator of the original equation to become 0 is called an **extraneous solution**.

To solve rational equations, you follow the following steps:

1. Factorize all the denominators and determine their LCM.
2. Restrict the values of the variable that make the LCM equal to 0.
3. Multiply both sides of the rational equation by the LCM and simplify.
4. Solve the resulting equation.
5. Check the answers against the restricted values in step 2. Any such value must be excluded from the solution.

Example 1 Solve each of the following equations:

a $\frac{2}{x+1}=\frac{3}{x-2}$

b $\frac{x}{x+4}-\frac{4}{x-4}=\frac{x^2+16}{x^2-16}$

c $\frac{10}{x(x-2)}+\frac{4}{x}=\frac{5}{x-2}$

d $\frac{3a-5}{a^2+4a+3}+\frac{2a+2}{a+3}=\frac{a-1}{a+1}$

Solution

a Your restrictions are $x \neq -1$ and $x \neq 2$. Now, multiply both sides of the equation by their LCM $(x+1)(x-2)$:

$$\frac{2}{x+1}=\frac{3}{x-2} \Rightarrow \left(\frac{2}{x+1}\right)\left(\frac{(x+1)(x-2)}{1}\right)=\left(\frac{3}{x-2}\right)\left(\frac{(x+1)(x-2)}{1}\right)$$

$$2(x-2)=3(x+1) \quad \Rightarrow \quad x=-7$$

This does not contradict our restrictions that $x \neq -1$ and $x \neq 2$.

Thus, our solution set is $\{-7\}$.

b Your restrictions are $x \neq -4$ and $x \neq 4$. Now multiply both sides by the LCM $(x-4)(x+4)$, which will get rid of the denominators:

$$x(x-4)-4(x+4)=x^2+16 \Rightarrow x^2-8x-16=x^2+16 \Rightarrow -8x=32 \Rightarrow x=-4$$

This is against our restriction $x \neq -4$, and must be excluded from our solution. Since there are no other values in our solution, the solution is $\varnothing$.

c The LCM here will be $x(x-2)$, and x cannot be 0 or 2. Multiplying both sides of the equation by this denominator:

$$\left(\frac{10}{x(x-2)}\right)\left(\frac{x(x-2)}{1}\right)+\left(\frac{4}{x}\right)\left(\frac{x(x-2)}{1}\right)=\left(\frac{5}{x-2}\right)\left(\frac{x(x-2)}{1}\right)$$

$$\Rightarrow 10+4(x-2)=5x \Rightarrow 10+4x-8=5x \Rightarrow 4x+2=5x$$

$$\Rightarrow x=2.$$

But $x=2$ is not allowed. Thus, the solution set is $\varnothing$.

d Since $a^2+4a+3=(a+3)(a+1)$, the LCM is $(a+3)(a+1)$, where a cannot be -3 or -1. Now you can multiply both sides by the LCM

$$\left(\frac{3a-5}{(a+3)(a+1)}\right)\left(\frac{(a+3)(a+1)}{1}\right)+\left(\frac{2a+2}{a+3}\right)\left(\frac{(a+3)(a+1)}{1}\right)=\left(\frac{a-1}{a+1}\right)\left(\frac{(a+3)(a+1)}{1}\right)$$

$$\Rightarrow 3a-5+(2a+2)(a+1)=(a-1)(a+3)$$

When simplified this gives $a^2+5a=0$ or $a(a+5)=0$.

This gives us $a = 0$ or $a = -5$.

These do not contradict our restrictions $a \neq -3$ and $a \neq -1$.

Thus, our solution set is $\{-5, 0\}$.

Example 2 One integer is four less than five times another. The sum of their reciprocals is $\frac{2}{3}$. What are the integers?

Solution When we encounter such word problems or other real life problems, we first assign variables to the unknowns. Now, let the unknown integers be x and y. Then one is four less than five times another can be written as $x = 5y - 4$, for x, y integers.

The sum of their reciprocals: $\frac{1}{x} + \frac{1}{y} = \frac{2}{3}$.

Substituting for x, we get: $\frac{1}{5y-4} + \frac{1}{y} = \frac{2}{3}$.

This rational equation reduces to the quadratic equation

$5y^2 - 13y + 6 = 0$, with solutions $y = 2$ and $y = \frac{3}{5}$.

But, since $\frac{3}{5}$ is not an integer, the only solution for y is 2. Solving for x, we get $x = 6$.

Thus the required integers are 2 and 6.

Exercise 2.6

1 State the universal set and solve each of the following rational equations:

a $\frac{3}{x+2} - \frac{1}{x} = \frac{1}{5x}$

b $\frac{x-6}{x} = \frac{x+4}{x} + 1$

c $\frac{4}{a} = \frac{1}{a^2+4a} - \frac{a+3}{a^2+4a}$

d $\frac{2}{x-4} - \frac{3}{x+1} = \frac{6}{x-1}$

e $\frac{3x-2}{5} = \frac{4x}{7}$

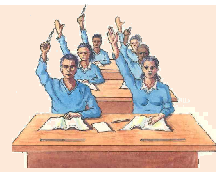
f $\frac{x+4}{x-5} - \frac{1}{x+5} = \frac{10}{x^2-25}$

2 Two planes leave an airport flying at the same rate. If the first plane flies 1.5 hours longer than the second plane and travels 2700 miles while the second plane travels only 2025 miles, for how long was each plane flying?

3 A tree casts a shadow of 34 feet at the time when a 3-foot tall child casts a shadow of 1.7 feet. What is the height of the tree?

2.3 RATIONAL FUNCTIONS AND THEIR GRAPHS

ACTIVITY 2.8



Identify the types (names) of each of the following functions.

State their domains.

a $f(x) = 3x + 5$ **b** $g(x) = 4 - x + 3x^2$

c $f(x) = log(x + 1)$ **d** $g(x) = 2^{3x+2}$

e $f(x) = 5\cos x$ **f** $g(x) = \sqrt{9 - x^2}$

2.3.1 Rational Functions

Definition 2.6

A **rational function** is a function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials in x and $q(x) \neq 0$.

Example 1 Which of the following are rational functions?

a $f(x) = \dfrac{x-2}{2x^3 + x^2 - x}$ **b** $g(x) = \dfrac{x^2 + 3x + 2}{1}$ **c** $h(x) = \sqrt{9 - x^2}$

Solution f and g are rational functions, while h is not.

$g(x) = \dfrac{x^2 + 3x + 2}{1}$ is the same as $y = x^2 + 3x + 2$, so, any polynomial function is a rational function.

Note:

a A rational function is said to be in lowest terms, if $p(x)$ and $q(x)$ have no common factor other than 1.

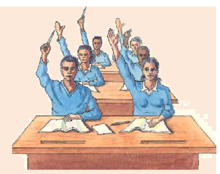
b The domain of a rational function f is the set of all real numbers except the values of x that make the denominator $q(x)$ zero.

Example 2 Give the domain of the function $f(x)=\dfrac{x+1}{2x^2+5x-3}$.

Solution The denominator $2x^2+5x-3=0 \Rightarrow (2x-1)(x+3)=0 \Rightarrow x=\frac{1}{2}$ or $x=-3$.

Thus, our domain is the set of all real numbers except $\frac{1}{2}$ and −3, because $\frac{1}{2}$ and −3 both make the denominator equal to 0.

ACTIVITY 2.9



Given the rational functions $f(x)=\dfrac{1}{x}$ and $g(x)=\dfrac{x}{x-2}$, find the following functional values and plot the corresponding points on the coordinate plane.

1 a $f(2)$ b $f(-3)$ c $f(0.4)$ d $f(-1.5)$

2 a $g(0)$ b $g(3)$ c $g(-2)$ d $g(2.5)$

Group Work 2.1

Do the following in groups.

Consider the function $f(x)=\dfrac{1}{x}$.

1 What is its domain?

2 a Fill in the following table for values of x to the left of 0:

x	−1	−0.5	−0.1	−0.01	−0.001	$\to 0$
$f(x)$						$\to -\infty$

b Fill in the following table for values of x to the right of 0:

x	1	0.5	0.1	0.01	0.001	$\to 0$
$f(x)$						$\to \infty$

3 Complete the following sentences:

As x approaches 0 from the left, $f(x)$ ____ without bound.

As x approaches 0 from the right, $f(x)$____ without bound.

As x increases or decreases without bound, the values of $f(x)=\dfrac{1}{x}$ approach ___

4 Here is the graph of $f(x)=\dfrac{1}{x}$.

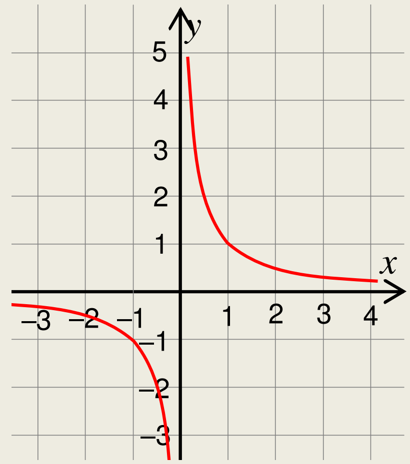


Figure 2.1

Do your observations correspond with the graph?

✍Note:

These two behaviours of f near $x = 0$ are denoted as follows.

a $f(x) \to -\infty$ as $x \to 0^-$

b $f(x) \to \infty$ as $x \to 0^+$

In this case, the line $x = 0$ (the y-axis) is called a **vertical asymptote** of the graph of f.

In addition, we have:

c $f(x) \to 0$ as $x \to -\infty$

d $f(x) \to 0$ as $x \to \infty$

Here, the line $y = 0$ (the x-axis) is called a **horizontal asymptote** of the graph of f.

Definition 2.7

1 The line $x = a$ is called a **vertical asymptote** of the graph of f if

$f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$, either from the left or from the right.

2 The line $y = b$ is called a **horizontal asymptote** of the graph of f if

$f(x) \to b$ as $x \to \infty$ or as $x \to -\infty$.

Rules for asymptotes and holes

Once the domain is established and the restrictions are identified, here are the pertinent facts.

Note:

Let $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{ax^n+...+a_0}{bx^m+...+b_0}$, be a rational function, where n is the largest exponent in the numerator and m is the largest exponent in the denominator.

1. The graph will have a vertical asymptote at $x = a$ if $q(a) = 0$ and $p(a) \neq 0$. In case $p(a) = q(a) = 0$, the function has either a hole at $x = a$ or requires further simplification to decide.
2. If $n < m$, then the x-axis is the horizontal asymptote.
3. If $n = m$, then the line $y=\dfrac{a}{b}$ is a horizontal asymptote.
4. If $n = m + 1$, the graph has an oblique asymptote and we can find it by long division.
5. If $n > m + 1$, the graph has neither an oblique nor a horizontal asymptote.

Example 3 Give the vertical and horizontal asymptotes, if they exist:

a $f(x)=\dfrac{1}{x+2}$ **b** $f(x)=\dfrac{x-2}{x^2-4}$

c $f(x)=\dfrac{x^2-1}{x^2+3x+2}$ **d** $f(x)=\dfrac{(x-1)(x+1)}{(x+1)^2(x+2)}$

Solution

a $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{1}{x+2}$. The domain of f is $\{x : x \neq -2\}$.

Since $p(-2) \neq 0$ and $q(-2) = 0$, $x = -2$ is a vertical asymptote.
Besides degree of $p(x)$ < degree of $q(x)$. Thus, $y = 0$ is a horizontal asymptote..

b Consider the rational function $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{x-2}{x^2-4}$. The domain of f is all real numbers except $x=-2$ and $x=2$.

$p(-2) \neq 0$ and $q(-2) = 0$. Thus, $x = -2$ is a vertical asymptote.

$p(2) = q(2) = 0$. Thus, f has a hole at $\left(2, \dfrac{1}{4}\right)$

Cancelling out the common factor $x - 2$, we obtain $g(x) = \dfrac{1}{x+2}$, and $g(2) = \dfrac{1}{4}$.

There $n = 1$ and $m = 2$. Therefore, the x-axis is a horizontal asymptote.

c Factorizing numerator and denominator gives:

$$f(x) = \frac{x^2 - 1}{x^2 + 3x + 2} = \frac{p(x)}{q(x)} = \frac{(x-1)(x+1)}{(x+1)(x+2)}$$

At $x = -1$, $p(-1) = q(-1) = 0$. Reducing to lowest terms we have:

$g(x) = \dfrac{x-1}{x+2}$, which gives $g(-1) = -2 \neq 0$. Thus, f has a hole at $(-1, -2)$.

At $x = -2$, $p(-2) = 3 \neq 0$ and $q(-2) = 0$. Thus $x = -2$ gives a vertical asymptote.

Since the degree of the numerator is equal to the degree of the denominator, $y = 1$ is a horizontal asymptote.

d $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{(x-1)(x+1)}{(x+1)^2(x+2)}$, $p(-2) \neq 0$ and $q(-2) = 0$. Thus $x = -2$ is a vertical asymptote. Again $p(-1) = q(-1) = 0$. However, after simplification by factorizing, we find that $x = -1$ is a vertical asymptote.

Since $n = 2 < 3 = m$, the x-axis is a horizontal asymptote.

Example 4 Find the oblique asymptote of the function $f(x) = \dfrac{x^2 + 1}{x - 1}$.

Solution Since the degree of the numerator is one more than that of the denominator, the graph of f has an oblique asymptote. Applying long division yields:

$$f(x) = \frac{x^2 + 1}{x - 1} = (x+1) + \frac{2}{x-1}.$$

Thus, the equation of the oblique asymptote is the quotient part of the answer which would be $y = x + 1$.

ACTIVITY 2.10

For each of the following rational functions, find the domain and identify the type of asymptotes.

a $f(x) = \dfrac{3}{x+4}$ **b** $f(x) = \dfrac{2x+1}{x}$ **c** $f(x) = \dfrac{x-3}{x^2-9}$

d $f(x) = \dfrac{x^2 - x}{x+1}$ **e** $f(x) = \dfrac{4x}{1-3x}$ **f** $f(x) = \dfrac{x^2 - x - 2}{x-1}$

The zeros of a rational function

Definition 2.8

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function. An element a in the domain of f is called a zero of f, if and only if $p(a) = 0$.

Example 5 Find the zeros of the following rational functions:

a $f(x) = \frac{x^2 + 3x + 2}{x^2 - 2x - 3}$ **b** $f(x) = \frac{x^2 - 6x + 9}{x^2 - 9}$

Solution

a We first factorize both numerator and denominator.

$f(x) = \frac{(x+1)(x+2)}{(x+1)(x-3)}$ The domain is $\mathbb{R} \setminus \{-1, 3\}$. Now for any x in the domain $f(x) = 0$ means the numerator $x^2 + 3x + 2 = 0$. i.e. $x = -1$ or $x = -2$. But, since $x = -1$ is not in the domain of f, the only zero of f is $x = -2$.

b Factorize both numerator and denominator: $f(x) = \frac{(x-3)^2}{(x+3)(x-3)}$

The domain is $\mathbb{R} \setminus \{-3, 3\}$. The numerator is zero at $x = 3$. But since 3 is not in the domain f, f has no zero.

2.3.2 Graphs of Rational Functions

In this subsection, you will use the zeros and asymptotes of rational functions to help draw their graphs.

Steps to sketch the graph of a rational function:

1. Reduce the rational function to lowest terms and check for any open holes in the graph.
2. Find x-intercept(s) by setting the numerator equal to zero.
3. Find the y-intercept (if there is one) by setting $x = 0$ in the function.
4. Find all its asymptotes (if any).
5. Determine the parity (i.e. whether it is even or odd or neither).

6 Use the x-intercepts and vertical asymptote(s) to divide the x-axis into intervals. Choose a test point in each interval to determine if the function is positive or negative there. This will tell you whether the graph approaches the vertical asymptote in an upward or downward direction.

7 Sketch the graph! Except for breaks at the vertical asymptotes or cusps, the graph should be a nice smooth curve with no sharp corners.

To draw the graph of $f(x)=\dfrac{p(x)}{q(x)}$,

We need to find	Criteria
Domain	$\mathbb{R}\setminus\{x: q(x)=0\}$
x - intercept	Zero of f
y - intercept	$x=0$ and $0\in$ domain of f
Vertical asymptote	$P(x)\neq 0$ and $q(x)=0$
Horizontal asymptote	Degree of $p(x)\leq$ Degree of $q(x)$
Oblique asymptote	Degree of $p(x)=$ Degree $q(x)+1$
Parity	f is odd or even or neither

Group Work 2.2

Do the following in groups. For each of the following functions, find the domain, x-intercept, the y-intercept, the asymptotes and the parity (if they exist). List them in tables.

a $f(x)=\dfrac{x+1}{(x-2)(x+3)^2}$ **b** $f(x)=\dfrac{x^2+5x+6}{x+1}$ **c** $f(x)=\dfrac{x-2}{x^2-4}$

Example 6 Sketch the graph of each of the following functions:

a $f(x)=-\dfrac{1}{x^2}$.

b $f(x)=\dfrac{3x^2}{(x-2)(x+1)}$

c $f(x)=\dfrac{x+1}{(x-2)(x+3)^2}$

d $f(x)=\dfrac{x^2+5x+6}{x+1}$.

e $f(x)=\dfrac{x-2}{x^2-4}$.

Solution

a The function $f(x) = -\frac{1}{x^2}$ cannot be reduced any further. This means that there will be no open holes on the graph of this function.

x - intercept	none
y - intercept	none
Vertical asymptote	$x = 0$
Horizontal asymptote	$y = 0$
Oblique asymptote	none
Parity	f is even

Next, we find and plot several other points on the graph.

x	-2	-1	1	2
$y = -\frac{1}{x^2}$	$-\frac{1}{4}$	-1	-1	$-\frac{1}{4}$

This table is called **a table of values**.

Finally, we draw curves through the points, approaching the asymptotes.

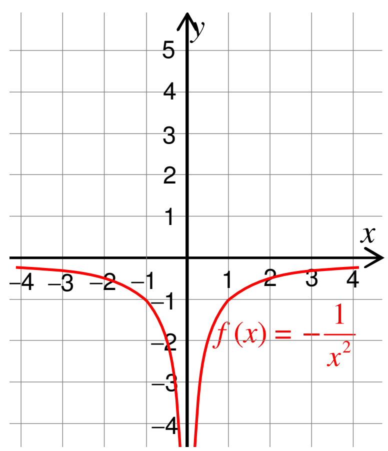


Figure 2.2

b The function $f(x) = \frac{3x^2}{(x-2)(x+1)}$ cannot be reduced any further. This means that there will be no open holes on the graph of this function.

x - intercept	$x = 0$ or $(0, 0)$
y - intercept	$y = 0$ or $(0, 0)$
Vertical asymptote	$x = -1$ and $x = 2$
Horizontal asymptote	$y = 3$ Note that the graph crosses the horizontal asymptote at $x = -2$.
Oblique asymptote	none
Parity	f is neither even nor odd. You can check this by taking a test point. For instance, $f(-4) \neq f(4)$ and $f(-4) \neq -f(4)$.

Next, we find and plot several other points on the graph.

x	-3	1	4	5
y	2.7	-1.5	4.8	4.17

Finally, we draw curves through the points, approaching the asymptotes.

Thus, the graph of f is:

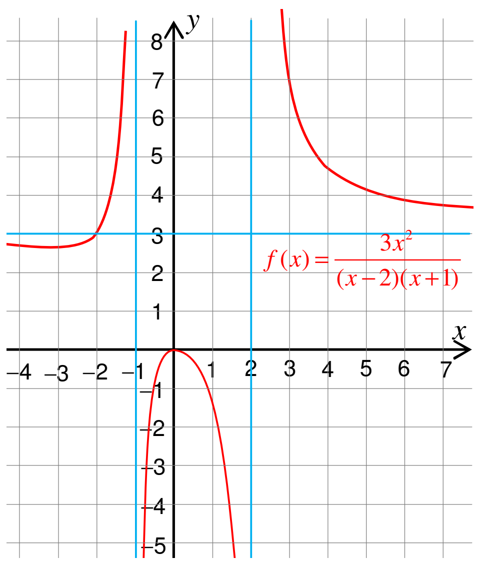


Figure 2.3

You have already found the necessary data to sketch the graphs of the functions in c, d, and e in Group Work 2.4. We only need to give the sketches of the graphs.

✍Note:

If $f(x)=\dfrac{p(x)}{q(x)}$ is in lowest terms and $(x-a)^n$ is a factor of $q(x)$, then

✓ the graph of f goes in opposite directions about the vertical asymptote $x = a$ when n is odd.

✓ the graph of f goes in the same direction about the vertical asymptote $x = a$ when n is even.

c

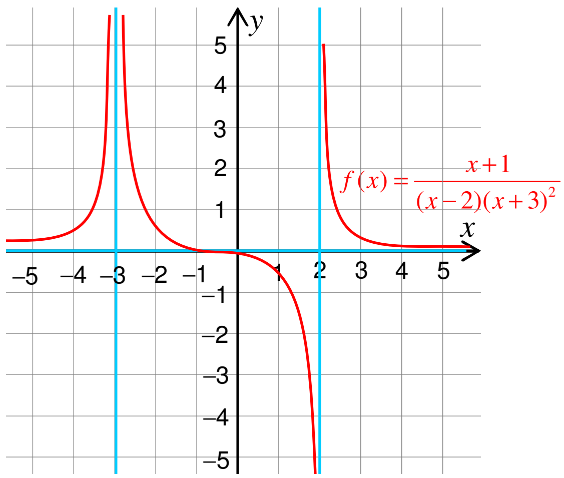


Figure 2.4

d

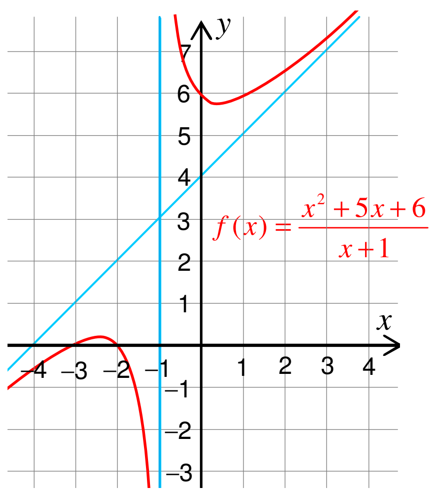


Figure 2.5

Observe that the graph of f approaches the line $y = x + 4$ as x approaches ∞ or as x approaches $-\infty$.

e

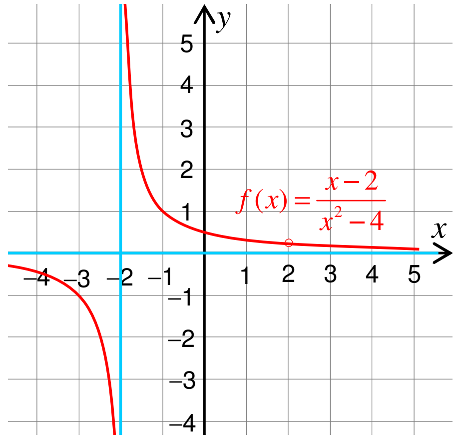


Figure 2.6

ACTIVITY 2.11

Using the graphs drawn in Example 6 above, solve each of the following inequalities.

a **i** $-\dfrac{1}{x^2} < 0$ **ii** $-\dfrac{1}{x^2} > 0$

b **i** $\dfrac{3x^2}{(x-2)(x+1)} < 0$ **ii** $\dfrac{3x^2}{(x-2)(x+1)} > 0$

c **i** $\dfrac{x+1}{(x-2)(x+3)^2} < 0$ **ii** $\dfrac{x+1}{(x-2)(x+3)^2} > 0$

d **i** $\dfrac{x^2+5x+6}{x+1} < 0$ **ii** $\dfrac{x^2+5x+6}{x+1} > 0$

e **i** $\dfrac{x-2}{x^2-4} < 0$ **ii** $\dfrac{x-2}{x^2-4} > 0$

Exercise 2.7

1 Sketch the graph of each of the following rational functions:

a $f(x) = \dfrac{x^2 - x - 12}{x^2 - 2x - 8}$ **b** $f(x) = \dfrac{3x^2 - 5x - 2}{x^2 - 1}$ **c** $f(x) = \dfrac{x^3 + 1}{x^2 - 1}$

d $f(x) = \dfrac{x^2 + 3x - 4}{x - 5}$ **e** $f(x) = \dfrac{2x^2 - 3x + 2}{x^2 + 1}$ **f** $f(x) = \dfrac{x + 2}{x^2 - 9}$

g $f(x) = \dfrac{-x}{x^2 + x - 2}$ **h** $f(x) = \dfrac{(x-1)(x+3)}{(x-2)}$ **i** $f(x) = \dfrac{x^2 - 9}{x - 3}$

2 For the rational functions in Question 1 b, d and h solve the inequalities $f(x) > 0$ and $f(x) < 0$ from their graphs.

Key Terms

domain	partial fractions
graphs of rational functions	rational equations
horizontal asymptote	rational expression
least common multiple	rational functions
oblique asymptote	vertical asymptote
operations on rational expressions	zeros of a rational function

Summary

1 A **rational expression** is the quotient of two polynomials.

2 Let $P(x), Q(x),$ and $R(x)$ be polynomials such that $Q(x) \neq 0$, then

$$\frac{P(x)}{Q(x)}+\frac{R(x)}{Q(x)}=\frac{P(x)+R(x)}{Q(x)} \quad \text{and} \quad \frac{P(x)}{Q(x)}-\frac{R(x)}{Q(x)}=\frac{P(x)-R(x)}{Q(x)}.$$

3 If $P(x), Q(x), R(x)$ and $S(x)$ are polynomials such that $Q(x) \neq 0$, $S(x) \neq 0$, then

$$\frac{P(x)}{Q(x)} \cdot \frac{R(x)}{S(x)}=\frac{P(x)R(x)}{Q(x)S(x)} \quad \text{and} \quad \frac{P(x)}{Q(x)} \div \frac{R(x)}{S(x)}=\frac{P(x)S(x)}{Q(x)R(x)} \text{ for } R(x) \neq 0$$

4 A **rational equation** is an equation where one or more of the terms are fractional ones.

5 A **rational function** is a function of the form $f(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials in x and $q(x) \neq 0$

6 The line $x = a$ is called a **vertical asymptote** of the graph of f if $f, (x) \to \pm\infty$ as $x \to a$ from the left or the right.

7 The line $y = b$ is called a **horizontal asymptote** of the graph of f, if $f(x) \to b$ as $x \to \pm\infty$.

8 An asymptote of the form $y = mx + b$, $m \neq 0$, is called an **oblique asymptote**.

9 A **zero** of $f(x)=\frac{p(x)}{q(x)}$ is a value a for which $p(a) = 0$ but $q(a) \neq 0$.

Review Exercises on Unit 2

1 Simplify each of the following rational expressions.

a $\dfrac{2x-4}{x^2+x-6}$ **b** $\dfrac{x^2-x-6}{x^2+3x+2}$

c $\dfrac{x^2-5x}{x^2-25}$ **d** $\dfrac{x^3+8x^2+24x+45}{x^4+3x^3-27x-81}$

2 Perform the indicated operations and simplify.

a $\dfrac{x+5}{2}+\dfrac{x-5}{2}$ **b** $\dfrac{2x^2}{x+9}-\dfrac{162}{x+9}$

c $\dfrac{\dfrac{2}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{1}{x^2-1}}$ **d** $\dfrac{x^2-1}{x^2+3x-4}\bullet\dfrac{x^2+x-12}{x^2+4x+3}$

e $\dfrac{x^2-25}{(x-5)^2}\div\dfrac{x^2+16}{(x+4)^2}$ **f** $\dfrac{x}{x^3-1}\div\left[2-\dfrac{1}{1+\dfrac{1}{x-2}}\right]$

3 Decompose the following rational expressions into partial fractions.

a $\dfrac{3}{x^2-3x}$ **b** $\dfrac{x+1}{x^2+4x+3}$ **c** $\dfrac{2x-3}{(x-1)^2}$

d $\dfrac{x+1}{x^3+x}$ **e** $\dfrac{x-1}{x^3+x^2}$ **f** $\dfrac{5x+1}{x^2(x^2+4)}$

4 State the domain and solve each of the following rational equations.

a $\dfrac{4}{x^2}=\dfrac{5}{x}-\dfrac{1}{x^2}$ **b** $\dfrac{x-6}{x}=\dfrac{x+4}{x}+1$

c $\dfrac{3}{y+3}+\dfrac{3y}{y+3}=1$ **d** $\dfrac{1}{y^2-3y}+\dfrac{1}{y-3}=\dfrac{3}{y^2-3y}$

5 State the domain and sketch the graph of each of the following rational functions. Find intercepts and asymptotes, if there are any.

a $f(x)=\dfrac{x-3}{x+2}$ **b** $g(x)=\dfrac{3}{(x-5)^2}$ **c** $f(x)=\dfrac{x^2}{x^2+1}$

d $g(x)=\dfrac{5x}{x^2-4}$ **e** $f(x)=x+\dfrac{1}{x^2}$ **f** $g(x)=\dfrac{2x^3}{x^2+1}$

Unit 3

COORDINATE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- *understand specific facts and principles about lines and circles.*
- *know basic concepts about conic sections.*
- *know methods and procedures for solving problems on conic sections.*

Main Contents

3.1 STRAIGHT LINE

3.2 CONIC SECTIONS

Key terms

Summary

Review Exercises

INTRODUCTION

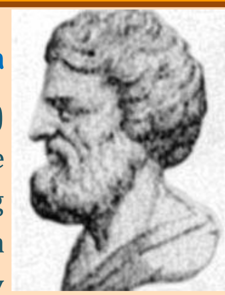
The method of analytic geometry reduces a problem in geometry to an algebraic problem by establishing a correspondence between a curve and a definite equation.

The concepts of lines and conics occur in nature and are used in many physical situations in nature, engineering and science. For instance, the earth's orbit around the sun is elliptical, while most satellite dishes are parabolic.

In this unit, you will study some more about straight lines and circles, and also the properties of the conic sections, *circle, parabola*, *ellipse* and *hyperbola*.

HISTORICAL NOTE

Apollonius of Perga



The Greek mathematician Apollonius (who died about 200 B.C.) studied conic sections. Apollonius is credited with providing the names "ellipse", "parabola", and "hyperbola" and for discovering that all the conic sections result from intersection of a cone and a plane. The theory was further advanced to its fullest form by Fermat, Descartes and Pascal during the 17^{th} century.

OPENING PROBLEM

A parabolic arch has dimensions as shown in the figure. Can you find the equation of the parabola? What are the respective values of y for $x = 5$, 10 and 15?

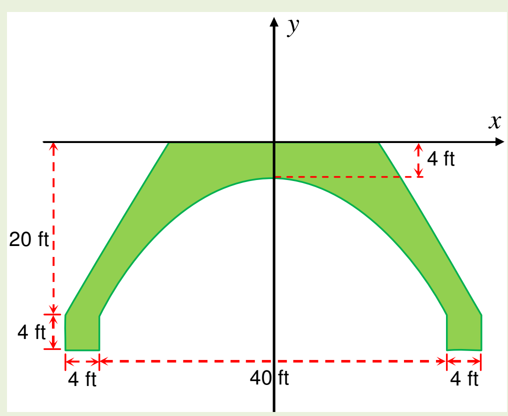


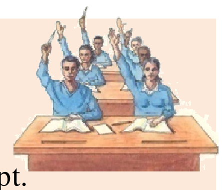
Figure 3.1

3.1 STRAIGHT LINE

Review on equation of a straight line

In Grade 10, you have learnt how to find the equation of a line and how to tell whether two lines are parallel or perpendicular by looking at their slopes. Now let us revise these concepts with the following Activity.

ACTIVITY 3.1



1 Given two points P (1, 4) and Q (3, –2), find the equation of a line passing through P and Q; and identify its slope and y-intercept.

2 Given the following equations of lines, characterize each line as vertical, horizontal or neither.

a $y = 3x - 5$ b $y = 7$ c $x = 2$ d $x + y = 0$

3 Identify each of the following pairs of lines as parallel, perpendicular or intersecting (but not perpendicular).

a $\ell_1 : y = 2x + 3;\ \ell_2 : y = \frac{1}{2}x - 2$

b $\ell_1 : y = 2x + 3;\ \ell_2 : y = -\frac{1}{2}x - 3$

c $\ell_1 : y = 2x + 3;\ \ell_2 : y = 2x + 5$

d $\ell_1 : 3x + 4y - 8 = 0\ \ell_2 : 4x - 3y - 9 = 0$

From the above Activity, you can summarize as follows.

- ✓ Any two points determine a straight line.
- ✓ If P(x_1, y_1) and Q(x_2, y_2) are points on a line with $x_1 \neq x_2$, then $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$ is the equation of the straight line and the ratio $m = \frac{y_2 - y_1}{x_2 - x_1}$ is the slope of the line.
- ✓ If $x_2 = x_1$, then the line is vertical and its equation is given by $x = x_1$; in this case the line has no slope.
- ✓ If two lines ℓ_1 and ℓ_2 have the same slope, then the two lines are parallel.

- ✓ If the product of the slopes of two lines ℓ_1 and ℓ_2 is –1, then the two lines are perpendicular.
- ✓ If the equation of a line is given by $y = mx + b$, then m is the slope of the line and b is its y - intercept.

Example 1 Find the equation of the line that passes through the points (–3, 2) and (4, 7) and identify its slope.

Solution The slope is given by $m = \dfrac{7-2}{4-(-3)} = \dfrac{5}{7}$

Thus, for any point P(x, y) on the line, $\dfrac{y-2}{x-(-3)} = \dfrac{5}{7} \Leftrightarrow y = \dfrac{5}{7}x + \dfrac{29}{7}$

3.1.1 Angle Between Two Lines on the Coordinate Plane

In the previous section, you have seen how to identify whether two lines are parallel or perpendicular. Now, when two lines are intersecting, you will see how to define the angle between the two lines and how to determine this angle.

Group Work 3.1

Consider the following graph and answer the questions that follow:

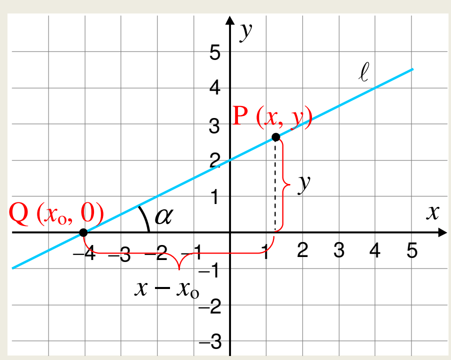


Figure 3.2

Find

a tan α

b slope of the line ℓ

c the relation between α and slope of ℓ

d If ℓ is vertical, then α = ____.

e If ℓ is horizontal, then $\alpha =$ ______.

f If $\alpha > 90°$, do you get the same relation as in **c** above between tan α and the slope of the line ℓ?

Definition 3.1

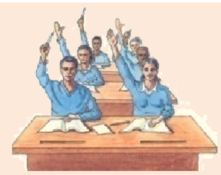
The angle α measured from the positive x-axis to a line in the counter-clockwise direction is called the **angle of inclination** of the line.

Example 2 If the angle of inclination of a line is 120°, then its slope is

$$\tan 120^{\circ} = -\sqrt{3}.$$

Example 3 If the slope of a line is 1, then its angle of inclination is 45°.

ACTIVITY 3.2



Consider the following two intersecting lines, and answer the questions that follow:

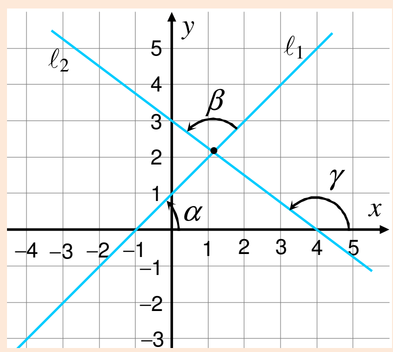


Figure 3.3

a What is the angle of inclination of ℓ_1?

b What is the angle of inclination of ℓ_2?

c Can you find any relation between α, γ and β?

Definition 3.2

The **angle between two intersecting lines** ℓ_1 and ℓ_2 is defined to be the angle β measured counter-clockwise from ℓ_1 to ℓ_2.

From the above Activity, you have $\beta = \gamma - \alpha$, slope of $\ell_1 = \tan\alpha$ and slope of $\ell_2 = \tan\gamma$

$$\text{Thus } \beta = \gamma - \alpha \Rightarrow \tan\beta = \tan(\gamma - \alpha) = \frac{\tan\gamma - \tan\alpha}{1 + \tan\gamma\tan\alpha}$$

Hence if m_1 is the slope of ℓ_1 and m_2 is the slope of ℓ_2, then the tangent of the angle between two lines ℓ_1 and ℓ_2 measured from ℓ_1 to ℓ_2 counter-clockwise is given by

$$\tan\beta = \frac{m_2 - m_1}{1 + m_1 m_2}, \text{ if } m_1 m_2 \neq -1.$$

So, the angle β can be found from the above equation.

✍Note:

The denominator $1 + m_1m_2 = 0 \Leftrightarrow m_1m_2 = -1 \Leftrightarrow \tan\beta$ is undefined $\Leftrightarrow \beta = 90^o$.

Thus, the angle between the two lines is $90^o \Leftrightarrow m_1m_2 = -1$ or $m_1 = -\dfrac{1}{m_2}$

Example 4 Given points P(2, 3), Q(–4, 1), C(2, 4) and D(6, 5), find the tangent of the angle between the line that passes through P and Q and the line that passes through C and D when measured from the line that passes through P and Q to the line that passes through C and D counter-clockwise.

Solution Let m_1 be the slope of the line through P and Q and m_2 be the slope of the line through C and D.

$$\text{Then, } m_1 = \frac{1-3}{-4-2} = \frac{-2}{-6} = \frac{1}{3} \text{ and } m_2 = \frac{5-4}{6-2} = \frac{1}{4}.$$

Thus, the tangent of the angle β between the line through P and Q and the line through C and D is

$$\tan\beta = \frac{m_2 - m_1}{1 + m_1m_2} = \frac{\frac{1}{4} - \frac{1}{3}}{1 + \frac{1}{3}\cdot\frac{1}{4}} = \frac{\frac{3-4}{12}}{\frac{12+1}{12}} = \frac{-1}{13}$$

Exercise 3.1

1 Write down the equation of the line that:

a passes through (–6, 2) and has slope $m = 4$

b passes through (6, 6) and (–1, 7)

c passes through (2, – 4) and is parallel to the line with equation $y = 7x - 10$.

d passes through (2, –4) and is perpendicular to the line with equation $y = 2x - 1$.

e passes through (1, 3) and the angle from the line with equation $y = x + 2$ to the line is 45°.

2 Find the tangent of the angle between the given lines.

a ℓ_1: $y = -3x + 2$; ℓ_2: $y = -x$ **b** ℓ_1 : $3x - y - 2 = 0$; ℓ_2: $4x - y - 6 = 0$

3 Determine B so that the line with equation $5x + By - 6 = 0$ is:

a parallel to the line with equation $y = \frac{4}{7}x + 1$

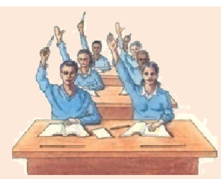
b perpendicular to the line with equation $y = \frac{4}{7}x + 1$

4 A car rental company leases automobiles for a charge of 20 Birr/day plus 2 Birr/km. Write an equation for the cost y Birr in terms of the distance x driven, if the car is leased for 5 days.

5 Water in a lake was polluted with sewage from a nearby town with $7mg$ of waste compounds per $1000l$ of water. It is determined that the pollution level would drop at the rate of 0.75 mg of waste compounds per 1000ℓ of water per year, if a plan proposed by environmentalists is followed. Let 2001 correspond to $x = 0$ and successive years correspond to $x = 1, 2, 3,$.... Find the equation $y = mx + b$ that helps predict the pollution level in future years, if the plan is implemented.

3.1.2 Distance between a Point and a Line on the Coordinate Plane

ACTIVITY 3.3



Given a line ℓ and a point P not on ℓ;

a Draw line segments from point P to ℓ. (as many as possible).

b Which line segment has the shortest length?

Definition 3.3

Suppose a line ℓ and a point P(x, y) are given. If P does not lie on ℓ, then we define the distance d from P to ℓ as the perpendicular distance between P and ℓ. If P is on ℓ, the distance is taken to be zero.

Let a line $\ell : Ax + By + C = 0$ with A, B and C all non-zero be given. To find the distance from the origin to the line $Ax + By + C = 0$, you can do the following:

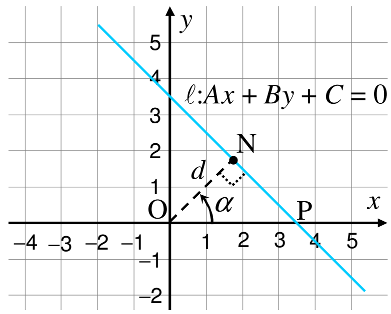


Figure 3.4

Draw $\overline{ON}$ perpendicular to $Ax + By + C = 0$. ΔONP is right angled triangle. Thus

$$|\cos\alpha| = \frac{d}{OP} \Rightarrow d = OP\,|\cos\alpha|.$$

The x-intercept of $Ax + By + C = 0$ is $-\frac{C}{A}$.

Thus, $d = \frac{|C|}{|A|}|\cos\alpha|$

Again $\overline{ON}$ being $\perp$ to the line $Ax + By + C = 0$ gives: slope of $\overline{ON}$ = $\tan\alpha = \frac{B}{A}$

$\left(\text{because slope of } Ax + By + C = 0 \text{ is } \frac{-A}{B}\right)$

This gives $|\cos\alpha| = \frac{|A|}{\sqrt{A^2 + B^2}}$

Hence, the distance from the origin to any line $Ax + By + C = 0$ with $A \neq 0$, $B \neq 0$ and $C \neq 0$ is given by $\frac{|C|}{\sqrt{A^2 + B^2}}$

✍Note:

The above formula is true when

i $C = 0$ (*in this case you get a line through the origin*) or

ii either $A = 0$ or $B = 0$ but not both, with $C \neq 0$ ($A = 0$ and $B \neq 0$ gives a horizontal line, while $A \neq 0$ and $B = 0$ gives a vertical line).

Example 5 Find the distance from the origin to the line $5x - 2y - 7 = 0$.

Solution The distance $d = \dfrac{|-7|}{\sqrt{5^2 + (-2)^2}} = \dfrac{7}{\sqrt{29}}$

Group Work 3.2

1 Consider a point P (h, k) on the xy-coordinate system. Form a new $x'y'$ coordinate system such that

- **a** the origin of the new system is at P (h, k)
- **b** the x'-axis is parallel to the x-axis and the y'-axis is parallel to the y-axis.

Let P be a point on the plane such that it has coordinates P (x, y) in the xy-system and P (x', y') in the $x'y'$-system. Express x' and y' in terms of x, y, h and k.

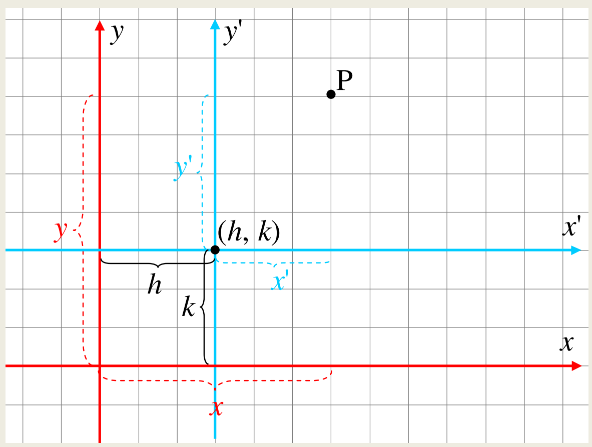


Figure 3.5

2 If $(h, k) = (3, 4)$, what is the representation of P$(-3, 2)$ (given in the xy-system) in the new $x'y'$-system?

From the above Group Work, you should get the **translation formulas:**

$$x' = x - h$$

$$y' = y - k$$

where (h, k) represents the origin of the new $x'y'$-system and (x', y') and (x, y) represent the coordinates of a point in the $x'y'$ and xy systems, respectively.

Example 6 Find the new coordinates of P$(5, -3)$, if the axes are translated to a new origin $(-2, -3)$.

Solution The formulae are $x' = x - h$ and $y' = y - k$. Here, $(h, k) = (-2, -3)$

Thus, the new coordinates of P(5, –3) are $x' = 5 - (-2) = 7$ and $y' = -3 - (-3) = 0$

Thus, in the $x'y'$-system, P(7, 0).

Next, we will find the distance between any point P(h, k) and a line

$\ell : Ax + By + C = 0.$

Translate the coordinate system to a new origin at P(h, k).

Let the equation of the line in the new x' y'-system be $A'x' + B'y' + C' = 0$. Then, the distance from P to ℓ is given by, $\dfrac{|C'|}{\sqrt{A'^2 + B'^2}}$

But, $A'x' + B'y' + C' = 0 \Leftrightarrow A'(x - h) + B'(y - k) + C' = 0$

$$A'x - A'h + B'y - B'k + C' = 0$$

$$A'x + B'y + (C' - A'h - B'k) = 0$$

Since in the xy - system the equation is $Ax + By + C = 0$

You get $A = A'$, $B = B'$, $C = C' - A'h - B'k$

So, $C' = A'h + B'k + C = Ah + Bk + C$

Hence the distance from P(h, k) to ℓ is given by $\dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$

Example 7 Find the distance between P(–4, 2) and $\ell : 2x + 9y - 3 = 0$

Solution $$d = \frac{|2(-4) + 9(2) - 3|}{\sqrt{2^2 + 9^2}} = \frac{|-8 + 18 - 3|}{\sqrt{85}} = \frac{7}{\sqrt{85}}$$

Exercise 3.2

1 Find the distance of each of the following lines from the origin.

a $4x - 3y = 10$ **b** $x - 5y + 2 = 0$

c $3x + y - 7 = 0$

2 Find the distance from each point to the given line.

a P(–3, 2) ; $5x + 4y - 3 = 0$ **b** P(4, 0) ; $2x - 3y - 2 = 0$

c P(–3, –5); $2x - 3y + 11 = 0$

3.2 CONIC SECTIONS

3.2.1 Cone and Sections of a Cone

The coordinate plane can be considered as a set of points which can be written as

$$\mathbb{R}^2 = \mathbb{R}\times\mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}.$$

If some of the points of the plane satisfy a certain condition, then these points make up a subset of the set of all points (i.e. the plane).

Definition 3.4

A **locus** is a system of points, lines or curves on a plane which satisfy one or more given conditions.

Example 1

The following are examples of loci (plural of locus).

1. The set $\{(x, y) \in \mathbb{R}^2 : y = 3x + 5\}$ is a line in the coordinate plane.
2. The set of all points on the x-axis which are at a distance of 3 units from the origin is $\{(-3, 0), (3, 0)\}$.

In this subsection, the plane curves called circles, parabolas, ellipses and hyperbolas will be considered.

Consider two right circular cones with common vertex and whose altitudes lie on the same line as shown in Figure 3.6.

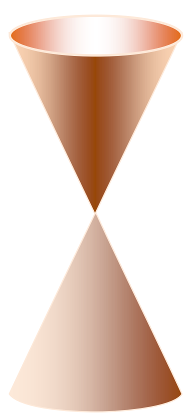


Figure 3.6

1 If a horizontal plane intersects /slices through one of the cones, the section formed is a circle.

2 If a slanted plane intersects /slices through one of the cones, then the section formed is either an ellipse or a parabola.

3 If a vertical plane intersects /slices through the pair of cones, then the section formed is a hyperbola.

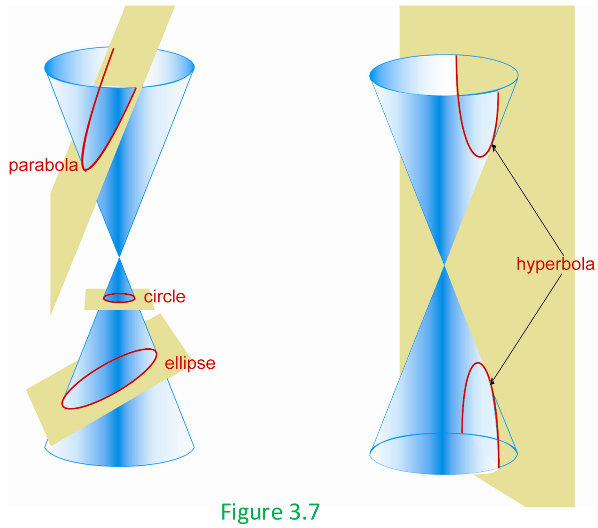


Figure 3.7

Since each of these plane curves are formed by intersecting a pair of cones with a plane, they are called **conic sections**.

3.2.2 Circles

ACTIVITY 3.4

Describe each of the following loci.

a The set of all points in a plane which are at a distance of 5 units from the origin.

b The set of all points in a plane which are at a distance of 4 units from the point $P(1, -2)$.

Each of the loci described in Activity 3.4 represents a circle.

Definition 3.5

A **circle** is the locus of a point that moves in a plane with a fixed distance from a fixed point. The **fixed distance** is called the **radius** of the circle and the **fixed point** is called the **centre** of the circle.

From the above definition, for any point P(x, y) on a circle with centre C (h, k) and radius r, PC = r and by the distance formula you have,

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

From this, by squaring both sides, you get

$$(x - h)^2 + (y - k)^2 = r^2$$

The above equation is called the **standard form of the equation of a circle**, with centre C (h, k) and radius r.

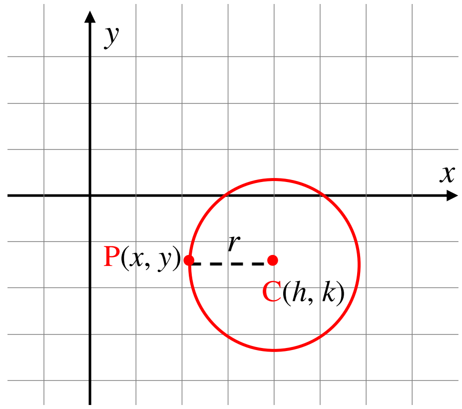


Figure 3.8

If the centre of a circle is at the origin (i.e. $h = 0$, $k = 0$), then the above equation becomes,

$$x^2 + y^2 = r^2$$

The above equation is called the **standard form of equation of a circle**, with centre at the origin and radius r.

Example 2 Write down the standard form of the equation of a circle with the given centre and radius.

a C (0, 0), $r = 8$ **b** C (2, –7), $r = 9$

Solution

a $h = k = 0$ and $r = 8$

Therefore, the equation of the circle is $(x - 0)^2 + (y - 0)^2 = 8^2$.

That is, $x^2 + y^2 = 64$.

b $h = 2$, $k = -7$ and $r = 9$.

Therefore, the equation of the circle is $(x - 2)^2 + (y + 7)^2 = 9^2$.

That is, $(x - 2)^2 + (y + 7)^2 = 81$.

Example 3 Write the standard form of the equation of the circle with centre at C (2, 3) and that passes through the point P (7, –3).

Solution Let r be the radius of the circle. Then the equation of the circle is

$$(x-2)^2 + (y-3)^2 = r^2$$

Since the point P (7, –3) is on the circle, you have

$$(7-2)^2 + (-3-3)^2 = r^2.$$

This implies, $5^2 + (-6)^2 = r^2$.

So, $r^2 = 61$.

Therefore, the equation of the circle is

$$(x-2)^2 + (y-3)^2 = 61.$$

Example 4 Give the centre and radius of the circle,

a $(x-5)^2 + (y+7)^2 = 64$ **b** $x^2 + y^2 + 6x - 8y = 0.$

Solution

a The equation is $(x-5)^2 + (y+7)^2 = 8^2$. Therefore, the centre C of the circle is C $(5, -7)$ and the radius r of the circle is $r = 8$.

b By completing the square method, the equation is equivalent to

$$x^2 + 6x + 9 + y^2 - 8y + 16 = 9 + 16 = 25.$$

This is equivalent to,

$$(x+3)^2 + (y-4)^2 = 5^2.$$

Therefore, the centre C of the circle is, C (–3, 4) and the radius r of the circle is $r = 5$.

ACTIVITY 3.5

1 Find the perpendicular distance from the centre of the circle with equation

$$(x-1)^2 + (y+4)^2 = 16$$

to each of the following lines with equations:

a $3x - 4y - 1 = 0$ **c** $3x - 4y + 2 = 0$

b $3x - 4y + 1 = 0$

2 Sketch the graph of the circle and each of the lines in Question 1 above, in the same coordinate system. What do you notice?

From Activity 3.5, you may have observed that:

1. If the perpendicular distance from the centre of a circle to a line is less than the radius of the circle, then the line intersects the circle at two points. Such a line is called a **secant** line to the circle.
2. If the perpendicular distance from the centre of a circle to a line is equal to the radius of the circle, then the line intersects the circle at only one point. Such a line is called a **tangent** line to the circle and the point of intersection is called the **point of tangency**.
3. If the perpendicular distance from the centre of a circle to a line is greater than the radius of the circle, then the line does not intersect the circle.

Note:

1. A line with equation $Ax + By + C = 0$ intersects a circle with equation $(x-h)^2 + (y-k)^2 = r^2$, if and only if,

$$\frac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}} \le r.$$

2. If a line with equation $Ax + By + C = 0$ intersects a circle with equation $(x-h)^2 + (y-k)^2 = r^2$, then $(x-h)^2 + \left(-\frac{A}{B}x - \frac{C}{B} - k\right)^2 = r^2$ is a quadratic equation in x. If $B = 0$, then $x = -\frac{C}{A}$ is a vertical line.

$$(y-k)^2 = r^2 - \left(-\frac{C}{A} - h\right)^2 = r^2 - \left(\frac{C + hA}{A}\right)^2,$$ which is a quadratic in y.

Solving this equation, you can get point(s) of intersection of the line and the circle.

Example 5 Find the intersection of the circle with equation $(x-1)^2 + (y+1)^2 = 25$ with each of the following lines.

a $4x - 3y - 7 = 0$ b $x = 4$

Solution

a $4x - 3y - 7 = 0 \Leftrightarrow y = \frac{4x - 7}{3}$

So $(x-1)^2 + \left(\frac{4x - 7}{3} + 1\right)^2 = 25$

$$\Rightarrow \quad (x-1)^2 + \left(\frac{4x - 4}{3}\right)^2 = 25$$

$$\Rightarrow \quad 9(x-1)^2 + (4x-4)^2 = 225$$

$$\Rightarrow \quad 9(x^2 - 2x + 1) + (16x^2 - 32x + 16) = 225$$

$$\Rightarrow \quad 9x^2 - 18x + 9 + 16x^2 - 32x + 16 = 225$$

$$\Rightarrow \quad 25x^2 - 50x - 200 = 0$$

$$\Rightarrow \quad x^2 - 2x - 8 = 0$$

$$\Rightarrow \quad (x+2)(x-4) = 0$$

$$\Rightarrow \quad x = -2 \text{ or } x = 4$$

This gives $y = -5$ and $y = 3$, respectively.

Hence the line and the circle intersect at the points P(–2, –5) and Q(4, 3).

b For the line $x = 4$,

$$\Rightarrow \quad (4-1)^2 + (y+1)^2 = 25$$

$$\Rightarrow \quad 9 + (y+1)^2 = 25$$

$$\Rightarrow \quad (y+1)^2 = 25 - 9 = 16$$

$$\Rightarrow \quad y + 1 = \pm 4$$

$$\Rightarrow \quad y = 3 \text{ or } y = -5.$$

Hence, the intersection points of the line and the circle are (4, 3) and (4, –5).

Example 6 For the circle $(x+1)^2 + (y-1)^2 = 13$, show that $y = \frac{3}{2}x - 4$ is a tangent line.

Solution The distance from C (–1, 1) to the line $-3x + 2y + 8 = 0$ is

$$d = \frac{|-3(-1) + 2(1) + 8|}{\sqrt{(-3)^2 + 2^2}} = \frac{|13|}{\sqrt{13}} = \sqrt{13} = r$$

Hence, $y = \frac{3}{2}x - 4$ is a tangent line to the circle

$$(x+1)^2 + (y-1)^2 = 13.$$

Example 7 Give the equation of the line tangent to the circle with equation $(x+1)^2 + (y-1)^2 = 13$ at the point P(–3, 4).

Solution First find the equation of the line ℓ that passes through the centre of the circle and the point of tangency.

The point of tangency is $T(x_o, y_o) = T(-3, 4)$ and the centre is $P(h, k) = P(-1, 1)$.

Therefore, equation of ℓ is given by:

$$\frac{y - y_0}{x - x_0} = \frac{y-4}{x-(-3)} = \frac{4-1}{-3+1}.$$

This implies, $\frac{y - 4}{x + 3} = -\frac{3}{2}$, which is equivalent to: $y-4=-\frac{3}{2}x - \frac{9}{2}$.

Hence $y=-\frac{3}{2}x - \frac{1}{2}$ is the equation of the line ℓ.

But the line ℓ is perpendicular to the tangent line to the circle at T(–3, 4).

Therefore, the equation of the tangent line is given by:

$$\frac{y-4}{x-(-3)} = \frac{2}{3} \Rightarrow \frac{y-4}{x+3} = \frac{2}{3}$$

Therefore $y = \frac{2}{3}x + 6$ is equation of the tangent line to the circle at (–3, 4).

Note:

✓ If a line ℓ is tangent to a circle $(x - h)^2 + (y - k)^2 = r^2$ at a point $T(x_o, y_o)$, then the equation of ℓ is given by

$$\frac{y-y_o}{x-x_o} = -\frac{x_o-h}{y_o-k}$$

Therefore, the equation of the tangent line to the circle in Example 7 can be found by:

$$\frac{y-y_o}{x-x_o} = \frac{y - 4}{x + 3} = -\left(\frac{-3+1}{4-1}\right) = \frac{2}{3}.$$

Example 8 Find the equation of the circle with centre at O(2, 5) and the line with equation $x-y=1$ is a tangent line to the circle.

Solution The distance from the centre O(2, 5) of the circle to the line with equation $x - y - 1 = 0$ is the radius.

Thus, $r = \frac{|2 - 5 - 1|}{\sqrt{1^2 + (-1)^2}} = 2\sqrt{2}$

Hence, the equation of the circle is $(x-2)^2 + (y-5)^2 = \left(2\sqrt{2}\right)^2 = 8$

Exercise 3.3

1 Write the standard form of the equation of a circle with the given centre and radius.

a C (–2, 3), $r = 5$ b C (8, 2), $r = \sqrt{2}$ c C (–2, –1), $r = 4$

2 Find the coordinates of the centre and the radius for each of the circles whose equations are given.

a $(x-2)^2 + (y-3)^2 = 7$ b $(x+7)^2 + (y+12)^2 = 36$

c $4(x+3)^2 + 4(y+2)^2 = 7$ d $(x-1)^2 + (y+3)^2 = 20$

e $x^2 + y^2 - 8x + 12y - 12 = 0$ f $x^2 + y^2 - 2x + 4y + 8 = 0$

3 Write the equation of the circle described below:

a It passes through the origin and has centre at (5, 2).

b It is tangent to the y-axis and has centre at (3, –4).

c The end points of its diameter are (–2, –3) and (4, 5).

4 A circle has centre at (5, 12) and is tangent to the line with equation $2x - y + 3 = 0$. Write the equation of the circle.

5 Find the equation of the tangent line to each circle at the indicated point.

a $x^2 + y^2 = 145$; P(9, –8) b $(x-2)^2 + (y-3)^2 = 10$; P(– 1, 2)

3.2.3 Parabolas

ACTIVITY 3.6

1 Draw the graph of each of the following functions.

a $y = x^2 + 2x + 3$ b $y = -x^2 + 5x - 4$

2 Find the axis of symmetry of the graphs in Question 1 above.

From Activity 3.6, you have seen that the graphs of both functions are parabolas; one opens upward and the other opens downward.

Definition 3.6

A **parabola** is the locus of points on a plane that have the same distance from a given point and a given line. The point is called the **focus** and the line is called the **directrix** of the parabola.

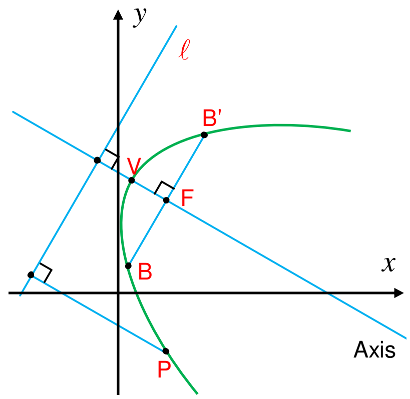
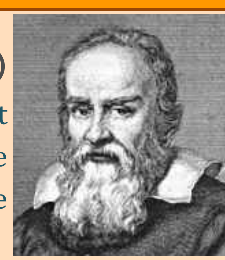


Figure 3.9

Consider Figure 3.9. Here are some terminologies for parabolas.

- ✓ F is the **focus** of the parabola.
- ✓ The line ℓ is the **directrix** of the parabola.
- ✓ The line which passes through the focus F and is perpendicular to the directrix ℓ is called the **axis** of the parabola.
- ✓ The point V on the parabola which lies on the axis of the parabola is called the **vertex** of the parabola.
- ✓ The chord $\overline{BB'}$ through the focus and perpendicular to the axis is called the **latus rectum** of the parabola.
- ✓ The distance $p = VF$ from the vertex to the focus is called the **focal length** of the parabola.

Historical Note



Galileo Galili (1564-1642)

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. More recently, parabolic shapes have been used in designing automobile highlights, reflecting telescopes and suspension bridges.

Now you are going to see how to find equation of a parabola with its axis of symmetry parallel to one of the coordinate axes. There are two cases to consider. The first case is when the axis of the parabola is parallel to the x-axis and the second case is when the axis of the parabola is parallel to the y-axis.

Equation of a parabola whose axis is parallel to the *x*-axis

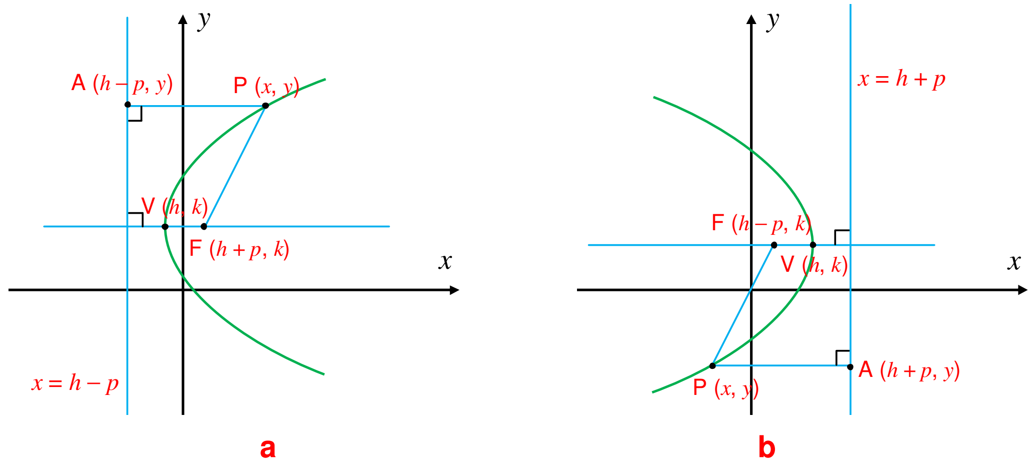


Figure 3.10

Let $V(h, k)$ be the vertex of the parabola. The axis of the parabola is the line $y = k$.

If the focus of the parabola is to the right of the vertex of the parabola, then the focus is $F(h + p, k)$ and the equation of the directrix is $x = h - p$. Let $P(x, y)$ be a point on the parabola. Then the distance from P to F is equal to the distance from P to the directrix. That is, $PF = PA$ where $A(h - p, y)$.

This implies $\sqrt{(x-(h+p))^2 + (y-k)^2} = \sqrt{(x-(h-p))^2 + (y-y)^2}$.

Squaring both sides gives you, $(x-(h+p))^2 + (y-k)^2 = (x-(h-p))^2$.

This implies, $x^2 - 2x(h+p) + (h+p)^2 + (y-k)^2 = x^2 - 2x\,(h-p) + (h-p)^2$.

This can be simplified to the form

$$(y-k)^2 = 4p(x-h)$$

This equation is called the **standard form of equation of a parabola** with vertex $V(h, k)$, focal length p; the focus F is to the right of the vertex and its axis is parallel to the x-axis. The parabola opens to the right.

If the focus of the parabola is to the left of the vertex of the parabola, then the focus is $F(h - p, k)$ and the equation of the directrix is $x = h + p$. With the same procedure as above, you can get the equation

$$(y-k)^2 = -4p\,(x-h)$$

This equation is called the **standard form of the equation of a parabola** with vertex $V(h, k)$, focal length p; the focus F is to the left of the vertex and its axis is parallel to the x-axis. In this case, the graph of the parabola opens to the left.

The standard form of the equation of a parabola with vertex V (h, k) and whose axis is parallel to x-axis is given below. Such a parabola is called an x-parabola.

✍Note:

The equation

$$(y-k)^2 = \pm 4p\,(x-h)$$

represents a parabola with:

- ✓ vertex V (h, k)
- ✓ focus $(h \pm p, k)$.
- ✓ directrix : $x = h \pm p$.
- ✓ axis of symmetry $y = k$.
- ✓ If the sign in front of p is positive, then the parabola opens to the right.
- ✓ If the sign in front of p is negative, then the parabola opens to the left.

Example 9 Find the equation of the directrix, the focus of the parabola, the length of the latus rectum and draw the graph of the parabola.

$$y^2 = 4x$$

Solution The vertex is at (0,0) and $4p = 4$. Hence $p = 1$.

The parabola opens to the right with focus $(h + p, k) = (0 + 1, 0) = (1, 0)$ and the directrix $x = h - p = 0 - 1 = -1$. The axis of the parabola is the x-axis.

The latus rectum passes through the focus F(1, 0) and is perpendicular to the axis, that is the x-axis.

Therefore, the equation of the line containing the latus rectum is $x = 1$.

To find the endpoints of the latus rectum, you have to find the intersection point of the line $x = 1$ and the parabola. That is, $y^2 = 4\times 1 = 4 \Leftrightarrow y = \pm 2$.

Therefore, the end points of the latus rectum are $(1, -2)$ and $(1, 2)$ and the length of the latus rectum is:

$$\sqrt{(1-1)^2 + (-2-2)^2} = \sqrt{16} = 4.$$

The graph of the parabola is given in Figure 3.11.

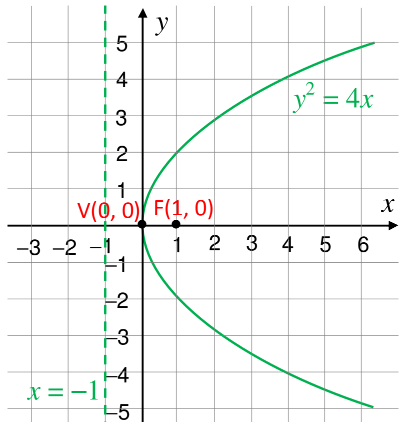


Figure 3.11

Example 10 Find the equation of the directrix and the focus of each parabola and draw the graph of each of the following parabolas.

a $4y^2 = -12x$ **b** $(y-2)^2 = 6(x-1)$ **c** $y^2 - 6y + 8x + 25 = 0$

Solution

a The equation $4y^2 = -12x$ can be written as $y^2 = \frac{-12x}{4} = -3x$

The vertex is V(h, k) = V(0, 0). $-4p = -3$ and $p = \frac{3}{4}$.

Since the sign in front of p is negative, the parabola opens to the left.

The directrix is $x = h + p = 0 + \frac{3}{4} = \frac{3}{4}$.

The focus is $F(h-p, k) = F\left(0 - \frac{3}{4}, 0\right) = F\left(-\frac{3}{4}, 0\right)$.

The graph of the parabola is given in Figure 3.12.

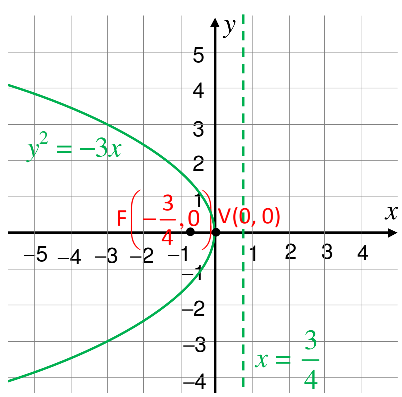


Figure 3.12

b The vertex is at V(*h*, *k*) = V(1, 2)

Since $4p = 6$, then $p = \frac{6}{4} = \frac{3}{2}$. The sign in front of *p* is positive. Hence the parabola opens to the right.

The focus is $F\,(h + p, k) = F\left(1 + \frac{3}{2}, 2\right) = F\left(\frac{5}{2}, 2\right)$

The directrix is $x = h - p = 1 - \frac{3}{2} = \frac{-1}{2}$. The axis of the parabola is the horizontal line $y = k$, i.e. $y = 2$ and the graph of the parabola is given in Figure 3.13.

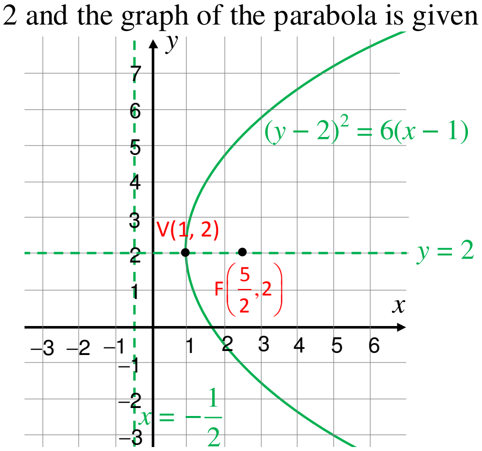


Figure 3.13

c By completing the square, the equation $y^2 - 6y + 8x + 25 = 0$ is equivalent to the equation $(y - 3)^2 = -8\,(x + 2)$. The vertex of the parabola is at V(*h*, *k*) = V(–2, 3) and $-4p = -8$ implies $p = 2$. The sign in front of *p* is negative. Hence the parabola opens to the left.

The focus $F(h - p, k) = F(-2\ -2{,}3) = F(-4, 3)$, the equation of the directrix is $x = h + p = -2 + 2 = 0$ and the equation of the axis of the parabola is $y = k$, i.e. $y = 3$ with its graph given in Figure 3.14.

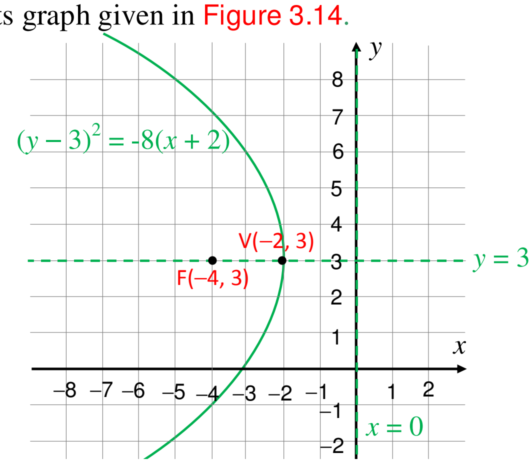


Figure 3.14

Example 11 Find the equation of the parabola with vertex V(–1, 4) and focus F(5, 4).

Solution Here V(h, k) = V(–1, 4).

Hence $h = -1$ and $k = 4$ and the focus is given by F($h + p$, k) = F(5, 4).

This implies $h + p = 5$ and $k = 4$. Then, $-1 + p = 5$, which implies $p = 6$

Since the focus F is to the right of the vertex V, the parabola opens to the right. Hence the equation of the parabola is given by:

$$(y - 4)^2 = 24(x + 1)$$

Equation of a parabola whose axis is parallel to the *y*-axis

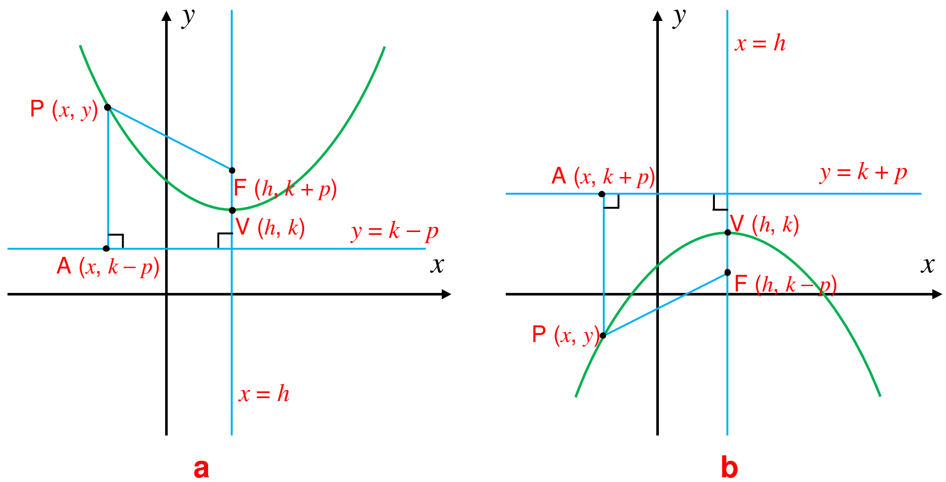


Figure 3.15

Let V(h, k) be the vertex of the parabola. The axis of the parabola is the line $x = h$.

If the focus of the parabola is above the vertex of the parabola, then the focus is F$(h, k + p)$ and the equation of the directrix is $y = k - p$. Let P(x, y) be a point on the parabola. Then the distance from P to F is equal to the distance from P to the directrix. That is, PF = PA where A$(x, k - p)$, as shown in Figure 3.15

This implies $\sqrt{(x-h)^2 + (y-(k+p))^2} = \sqrt{(x-x)^2 + (y-(k-p))^2}$.

This can be simplified to the form

$$(x-h)^2 = 4p\,(y-k)$$

The standard form of equation of a parabola with vertex V (h, k) and whose axis is parallel to the y-axis. Such a parabola is called a y-parabola.

✍Note:

The equation

$$(x-h)^2 = \pm 4p\,(y-k)$$

represents a parabola with

- ✓ vertex V (h, k)
- ✓ focus $F(h, k \pm p)$.
- ✓ directrix : $y = k \mp p$.
- ✓ axis of symmetry $x = h$
- ✓ If the sign in front of p is positive, then the parabola opens upward.
- ✓ If the sign in front of p is negative, then the parabola opens downward.

Example12 Find the vertex, focus and directrix of the following parabolas; sketch the graphs of the parabolas in *b* and *c*.

a $x^2 = 16y$ **b** $-2x^2 = 8y$

c $(x-2)^2 = 8(y+1)$ **d** $x^2 + 12y - 2x - 11 = 0$

Solution

a Here $4p = 16$ implies $p = 4$.

Since the sign in front of p is positive, the parabola opens upward.

The vertex is $V(h, k) = V(0, 0)$.

The focus is $F(0, p) = F(0, 4)$.

The directrix is $y = k - p = 0 - 4 = -4$.

b $-2x^2 = 8y$ can be written as $x^2 = -4y$.

Here, $-4p = -4$ implies $p = 1$.

Since the sign in front of p is negative, the parabola opens downward as shown in Figure 3.16.

The vertex is $V(h, k) = V(0, 0)$.

The focus is $F(h, k-p) = F(0, 0-1) = F(0, -1)$.

The directrix is $y = k + p = 0 + 1 = 1$.

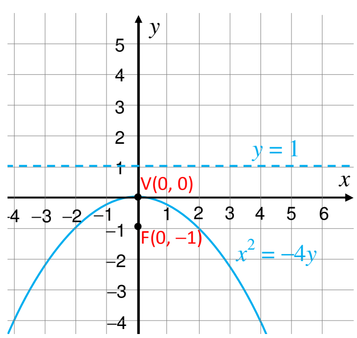


Figure 3.16

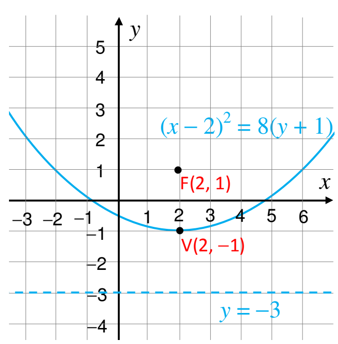


Figure 3.17

c Here $4p = 8$ implies $p = 2$.

Since the sign in front of p is positive, the parabola opens upward as shown in Figure 3.17.

The vertex

$$V(h, k) = V(2, -1).$$

The focus is

$$F(h, k+p) = F(2, -1+2) = F(2, 1).$$

The directrix is $y = k - p = -1 - 2 = -3$.

d The equation $x^2 + 12y - 2x - 11 = 0$ is equivalent to $(x-1)^2 = -12(y-1)$.

Hence $-4p = -12$ implies $p = 3$;

Since the sign in front of p is negative, the parabola opens downward.

The vertex is $V(h, k) = V(1, 1)$

The focus is $F(h, k-p) = F(1, 1-3) = F(1, -2)$

The directrix is $y = k + p = 1 + 3 = 4$

Example 13 (*Parabolic reflector*)

A paraboloid is formed by revolving a parabola about its axis. A spotlight in the form of a paraboloid 6 inches deep has its focus 3 inches from the vertex. Find the radius r of the opening of the spotlight.

Solution First locate a parabolic cross section containing the axis in a coordinate system and label all the known parts and parts to be found as shown in Figure 3.18.

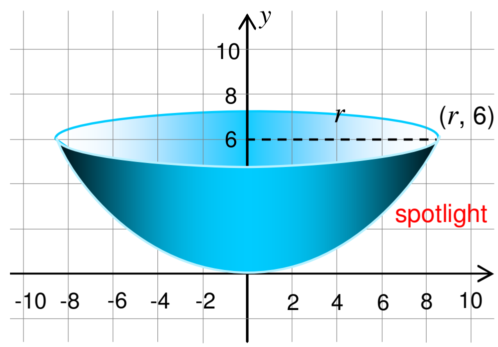


Figure 3.18

The parabola has the y-axis as its axis and the origin as its vertex. Hence the equation of the parabola is:

$$x^2 = 4py.$$

The focus is given $F(0, 3) = F(0, p)$ Thus $p = 3$ and the equation of the parabola is:

$$x^2 = 12y.$$

The point $(r, 6)$ is on the parabola.

$$\Rightarrow r^2 = 12 \times 6$$

$$\Rightarrow r^2 = 72$$

$$\Rightarrow r = \sqrt{72} \cong 8.49 \text{ inches.}$$

Exercise 3.4

1 Write the equation of each parabola given below.

a Vertex (–2, 5); focus (–2, –8) **b** Vertex (–3, 4); focus (–3, 12)

c Vertex (4, 6); focus (–8, 6) **d** Vertex (–1, 8); focus (6, 8)

2 Name the vertex, focus and directrix of the parabola whose equation is given and sketch the graph of each of the following.

a $x^2 = 2y$ **b** $(x + 2)^2 = 4(y - 6)$

c $(y + 2)^2 = -16(x - 3)$ **d** $(x - 3)^2 = 4y$

3 Write the equation of each parabola described below.

a Focus (3, 5); directrix $y = 3$ **b** Vertex (–2, 1); axis $y = 1$; $p = 1$

c Vertex (4, 3); passes through (5, 2), vertical axis

d Focus (5, 0); $p = 4$; vertical axis

4 Write the equation of each parabola described below.

a Vertex at the origin, axis along the x-axis, passing through A (3, 6)

b Vertex at (4, 2), axis parallel to the x-axis, passing through A (8, 7)

c Vertex at (5, –3), axis parallel to the y-axis, passing through B (1, 2)

5 The parabola has a multitude of scientific applications. A reflecting telescope is designed by using the property of a parabola:

If the axis of a parabolic mirror is pointed toward a star, the rays from the star, upon striking the mirror, will be reflected to the focus.

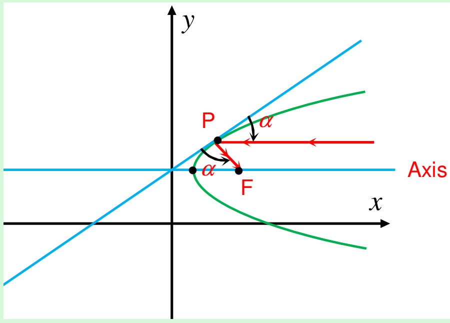


Figure 3.19

Answer the following questions

a A parabolic reflector is designed so that its diameter is 12 m when its depth is 4 m. Locate the focus.

b A parabolic head light lamp is designed in such a way that when it is 16 cm wide it has 6 cm depth. How wide is it at the focus?

6 Find the equation of the parabola determined by the given data.

a The vertex is at (1,2), the axis is parallel to the x-axis and the parabola passes through (6,3).

b The focus is at (3,4), the directrix is at $x = 8$.

3.2.4 Ellipses

Group Work 3.3

Do the following in groups.

1 Draw a circle of radius 5 cm.

2 Using two drawing pins, a length of a string and a pencil do the following. Push the pins into a paper at two points. Tie the string into a loose loop around the two pins. Pull the loop taut with the pencil's tip so as to form a triangle. Move the pencil around while keeping the string taut.

3 What do you observe from the two drawings?

Definition 3.7

An **ellipse** is the locus of all points in the plane such that the sum of the distances from two given fixed points in the plane, called the **foci**, is constant.

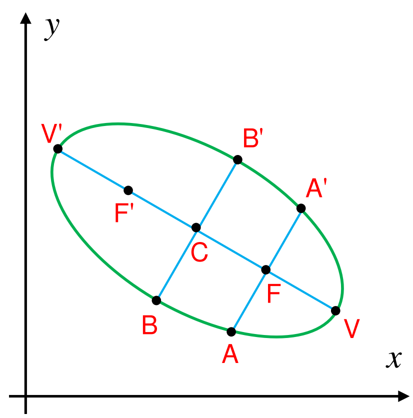


Figure 3.20

Consider Figure 3.20. Here are some terminologies for ellipses.

- ✓ F and F' are **foci**.
- ✓ V, V', B and B' are called **vertices** of the ellipse.
- ✓ $\overline{V'V}$ is called **the major axis** and $\overline{B'B}$ is called the **minor axis**.
- ✓ C, which is the intersection point of the major and minor axes is called the **centre** of the ellipse.

- ✓ $\overline{CV}$ and $\overline{CV'}$ are called **semi-major axes** and $\overline{CB}$ and $\overline{CB'}$ are called **semi-minor axes.**
- ✓ Chord $\overline{AA'}$ which is perpendicular to the major axis at F is called the **latus rectum** of the ellipse.
- ✓ The distance from the centre to a focus is denoted by c.
- ✓ The length of the **semi-major** axis is denoted by a and the length of the

  semi-minor axis is denoted by b.
- ✓ The eccentricity of an ellipse, usually denoted by e, is the ratio of the distance between the two foci to the length of the major axis, that is,

$$e = \frac{\text{distance between the two foci}}{\text{lengh of the major axis}} = \frac{c}{a}$$

which is a number between 0 and 1.

Note that V'F' = VF and VF + VF' = VV' = 2a, according to the definition. If P is any point on the ellipse, you have,

$$PF + PF' = 2a$$

Since B is on the ellipse, you also have that BF + BF' = 2a. But BF = BF'. This implies BF = a. By using Pythagoras Theorem for right angled triangle ΔBCF, you get,

$$CB^2 + CF^2 = BF^2$$

But CB = b, CF = c and BF = a. Therefore a, b and c have the relation,

$$b^2 + c^2 = a^2$$

Historical Note

Johannes Kepler (1571-1630)

In the 17th century, Johannes Kepler discovered that the orbits along which the planets travel around the Sun are ellipses with the Sun at one focus, (*his first law of planetary motion*).

Equation of an ellipse whose centre is at the origin

There are two cases to consider.

One of these cases is where the major axis of the ellipse is parallel to the x-axis as shown in Figure 3.21 below.

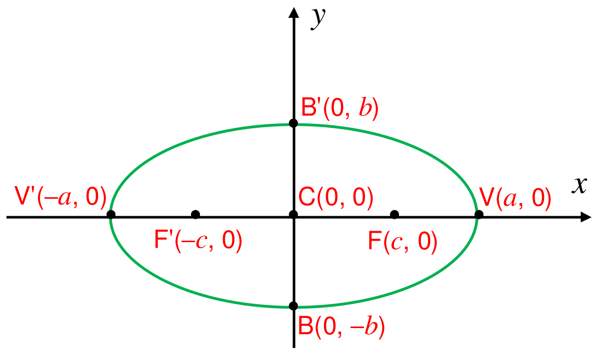


Figure 3.21

From the discussion so far, you have,

$$\text{PF}' + \text{PF} = 2a.$$

This implies $\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$

$$\Rightarrow \sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2}$$

Squaring both sides gives you,

$$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$$

Thus, $4a\sqrt{(x-c)^2+y^2} = 4a^2+(x-c)^2-(x+c)^2$

This implies $4a\sqrt{(x-c)^2+y^2} = 4a^2+x^2-2xc+c^2-x^2-2xc-c^2$

This gives you the result $a\sqrt{(x-c)^2+y^2} = a^2-cx$

Squaring both sides gives

$$a^2((x-c)^2+y^2) = (a^2-cx)^2$$

$$\Rightarrow a^2(x^2-2xc+c^2+y^2) = a^4-2a^2cx+c^2x^2$$

$$\Rightarrow a^2x^2-2a^2cx+a^2c^2+a^2y^2 = a^4-2a^2cx+c^2x^2$$

$$\Rightarrow (a^2-c^2)x^2+a^2y^2 = a^4-a^2c^2$$

$$\Rightarrow (a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2)$$

From the relation $a^2 = b^2 + c^2$, you get, $a^2 - c^2 = b^2$.

This gives you,

$$b^2x^2 + a^2y^2 = a^2b^2$$

By dividing both sides by $a^2 b^2$, you have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This equation is called the **standard form of an equation of an ellipse** whose major axis is horizontal and centre is at (0, 0).

Example 14 Give the coordinates of the foci of the ellipse shown below. Give the equation of the ellipse and find the eccentricity of the ellipse.

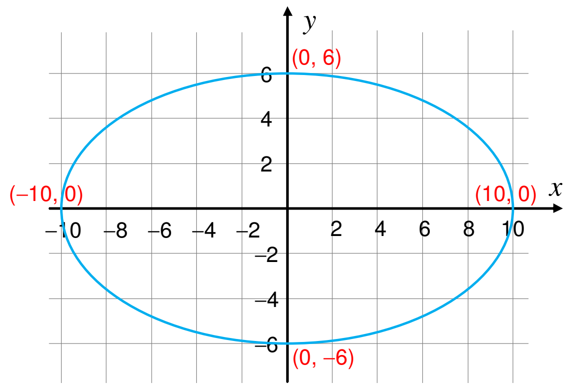


Figure 3.22

Solution From the graph observe that a = 10, and b = 6. Since $a^2 = b^2 + c^2$, then $100 = 36 + c^2$. Hence $c^2 = 64$. This implies c = 8.

Therefore, the centre is C (0, 0) and the foci are F'(–c, 0) = F'(–8, 0) and F(c, 0) = F(8, 0) since the major axis is horizontal.

Then the equation of the ellipse is $\dfrac{x^2}{100}+\dfrac{y^2}{36}=1$.

The eccentricity of the ellipse is $e=\dfrac{c}{a}=\dfrac{8}{10}=0.8$

Example 15 Find the equation of the ellipse with foci F'(–2, 0) and F (2, 0), a = 7.

Solution F'(–2, 0) and F (2, 0), implies that C (0, 0) and c = 2. The major axis of ellipse is horizontal.

From the relation $a^2 = b^2 + c^2$, you get $b^2 = a^2 - c^2 = 7^2 - 2^2 = 45$.

Hence, the equation of the ellipse is, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ or $\dfrac{x^2}{49}+\dfrac{y^2}{45}=1$.

Equation of an ellipse whose centre is C (*h*, *k*) different from the origin

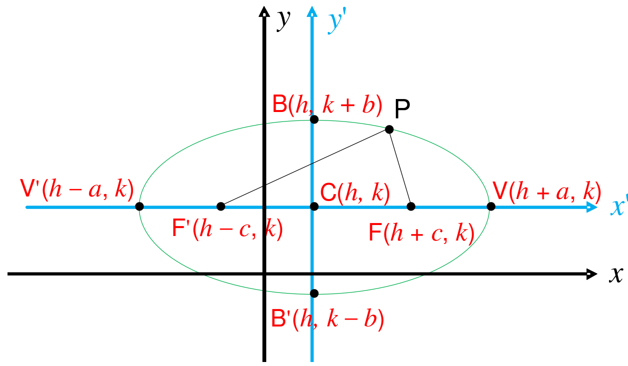


Figure 3.23

Let C(h, k) be the centre of the ellipse. Construct a new x' y'-coordinate system with origin at C(h, k). Then, for any point P on the ellipse with coordinates (x, y) in the xy-coordinate system and (x', y') in the new $x'y'$-coordinate system,

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

But then from translation formulae you have $x' = x - h$ and $y' = y - k$, which gives

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

which is the standard equation of an ellipse with centre at C(h, k) and major axis parallel to the x-axis.

Similarly, when the major axis is vertical, the standard equation of the ellipse is given by:

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, \text{ when C } (0\ ,0) \text{ and } \frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1, \text{ when C}(h, k)$$

Example 16 Find the coordinates of the centre, foci, the length of the major and minor axes, draw the graph of the ellipse, find the eccentricity of the ellipse and the length of the latus rectum.

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{1} = 1$$

Solution The centre of the ellipse is C (2, 1) and the major axis is horizontal. Also $a^2 = 9$ and $b^2 = 1$, which implies $a = 3$ and $b = 1$. Then the length of the major axis is 6 and the length of the minor axis is 2. Hence the vertices are V'(–1, 1), V (5, 1), B'(2, 0) and B (2, 2).

From the relation $c^2 = a^2 - b^2$, you get c = $2\sqrt{2}$ and the foci are $F(2-2\sqrt{2},\ 1)$ and $F'(2+2\sqrt{2},\ 1)$.

The eccentricity e of the ellipse is $e = \frac{c}{a} = \frac{2\sqrt{2}}{3}$.

The lines containing the latus rectums are vertical lines. These lines are $x = 2+\sqrt{2}$ and $x = 2-\sqrt{2}$. The intersection points of the line $x = 2+\sqrt{2}$ and the ellipse are given by:

$$\frac{(2+\sqrt{2}-2)^2}{9} + \frac{(y-1)^2}{1} = 1.$$

Solving this gives you: $y = \frac{3 \pm \sqrt{7}}{3}$.

Hence, the end points of one of the latus rectums are:

$$\left(2+\sqrt{2}, \frac{3\pm\sqrt{7}}{3}\right).$$

Therefore, the length of the latus rectum is $\frac{2\sqrt{7}}{3}$.

The graph of the ellipse is given in Figure 3.24.

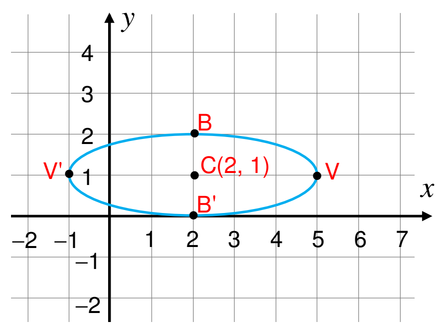


Figure 3.24

Example 17 Find the coordinates of the centre, foci, the length of the major and minor axes, draw the graph of the ellipse .

$$\frac{(y+2)^2}{25} + \frac{(x+2)^2}{16} = 1$$

Solution The centre of the ellipse is C (–2, –2) and the major axis is vertical. Also $a^2 = 25$ and $b^2 = 16$, which implies $a = 5$ and $b = 4$. So the length of the major axis is 10 and the length of the minor axis is 8 and also $c = \sqrt{a^2 - b^2} = 3$.

Therefore the foci are $(h, k \pm c) = (-2, -2 \pm 3)$, that is, $F'(-2, -5)$, $F(-2, 1)$ and also the vertices are $V'(-2, -7)$, $V(-2, 3)$, $B'(-6, -2)$, and $B(2, -2)$.

The graph of the ellipse is given in Figure 3.25.

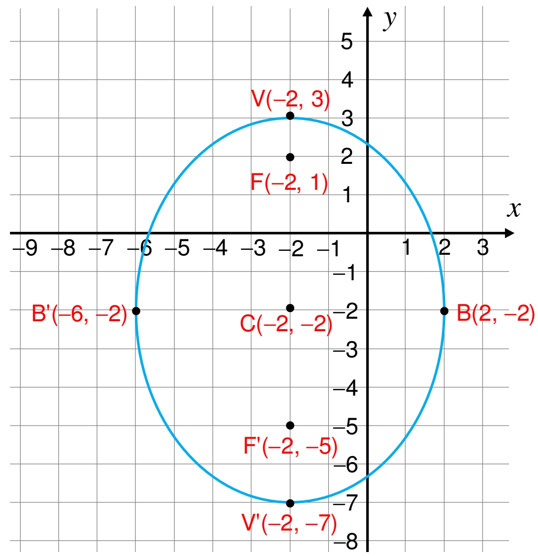


Figure 3.25

Exercise 3.5

1 Write the equation of each ellipse described below.

a C (0, 0); $a = 6$, $b = 4$; horizontal major axis

b Foci (–3, 0), (3, 0); $a = 8$

c C (0, 0); $a = 8$, $b = 6$; vertical major axis

d C (5, 0); $a = 5$, $b = 2$; horizontal major axis

2 Name the centre, the foci and the vertices of each ellipse whose equation is given. Also sketch the graph of each ellipse.

a $\dfrac{(x-3)^2}{25} + \dfrac{(y-4)^2}{16} = 1$ **b** $\dfrac{(y+2)^2}{25} + \dfrac{(x-1)^2}{4} = 1$

c $\dfrac{(y-2)^2}{25} + \dfrac{(x-3)^2}{5} = 1$

3 Find the equation of the ellipse with

a centre at (1, 4) and vertices at (10, 4) and (1, 2)

b foci at (–1, 0), (1, 0) and the length of the major axis 6 units.

c vertex at (6, 0), focus at (–1, 0) and centre at (0, 0).

d centre at $\left(0\ ,\ \frac{-1}{2}\right)$, focus at (0, 1) and passing through (2, 2).

e centre (0, 0), vertex (0, –5) and length of minor axis 8 units.

4 The planet Mars travels around the Sun in an ellipse whose equation is approximately given by

$$\frac{x^2}{(228)^2} + \frac{y^2}{(227)^2} = 1$$

where x and y are measured in millions of kilometres . Find

a the distance from the Sun to the other focus of the ellipse(*in millions of kilometres*).

b how close Mars gets to the Sun.

c the greatest possible distance between Mars and the Sun.

3.2.5 Hyperbolas

Definition 3.8

A **hyperbola** is defined as the locus of points in the plane such that the difference between the distances from two fixed points is a constant. The fixed points are called **foci**. The point midway between the foci is called the **centre** of the hyperbola.

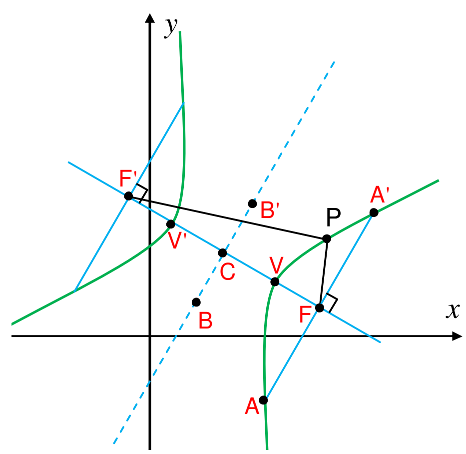


Figure 3.26

Consider Figure 3.26. Here are some terminologies for hyperbolas.

✓ F and F' are the **foci** of the hyperbola.

✓ C is the **centre** of the hyperbola.

- ✓ The points V and V' on each branch of the hyperbola nearest to the centre are called **vertices**.
- ✓ $\overline{V'V}$ is called the **transverse axis** of the hyperbola and CV = CV' is denoted by a and CF = CF' is denoted by c.
- ✓ Denote $c^2 - a^2$ by b^2, so that $b = \sqrt{c^2 - a^2}$.
- ✓ The segment of symmetry perpendicular to the transverse axis at the centre, which has length $2b$, is called the **conjugate axis**.
- ✓ The end points B and B' of the **conjugate axis** of the hyperbola are called co-vertices.
- ✓ The **eccentricity** of the hyperbola, usually denoted by e, is the ratio of the distance between the two foci to the length of the transverse axis, that is,

$$e = \frac{\text{distance between the two foci}}{\text{lengh of the transverse axis}} = \frac{c}{a}$$

  which is a number greater than 1.
- ✓ The chords with end points on the hyperbola passing through the foci and perpendicular to $\overline{FF'}$ are called the **latus rectums**.

✍Note:

Hyperbolas occur frequently as graphs of equations in Chemistry, Physics, Biology and Economics (Boyle's Law, Ohm's Law, supply and demand curves).

Equation of a hyperbola with centre at the origin and whose transverse axis is horizontal

Consider a hyperbola with foci F'($-c$, 0), F(c, 0) and centre C (0, 0).

Then, a point P(x, y) is on the hyperbola, if and only if

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a$$

Adding $\sqrt{(x+c)^2+y^2}$ to both sides of the above equation gives you

$$\sqrt{(x-c)^2+y^2} = \pm\, 2a + \sqrt{(x+c)^2+y^2}\,.$$

By squaring both sides you have,

$$(x-c)^2+y^2 = 4a^2 \pm\, 4a\sqrt{(x+c)^2+y^2} + (x+c)^2 + y^2.$$

This implies $\pm 4a\sqrt{(x+c)^2+y^2} = 4a^2 + x^2 + 2xc + c^2 - x^2 + 2xc - c^2$

That is $\pm 4a\sqrt{(x+c)^2+y^2} = 4a^2+4xc$.

This implies, $\pm a\sqrt{(x+c)^2+y^2} = a^2+xc$.

Again squaring both sides of the above equation gives you:

$$a^2\ ((x+c)^2+y^2) = a^4+2a^2xc + x^2c^2$$

This implies, $(a^2 - c^2)\ x^2 + a^2 y^2 = a^2\ (a^2 - c^2)$.

Recall that $c^2 - a^2 = b^2$. Thus, $-b^2\ x^2 + a^2\ y^2 = -a^2\ b^2$, which reduces to

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This equation is called the **standard form of equation of a hyperbola** with centre at C(0, 0) and transverse axis horizontal.

Example 18 Find the equation of a hyperbola, if the foci are F (2, 5) and F'(–4, 5) and the transverse axis is 4 units long. Draw the graph of the hyperbola.

Solution The mid-point of $\overline{FF'}$ is the centre of the hyperbola and it is C (–1, 5). The transverse axis is $2a = 4$. So, $a = 2$ and FF' = $2c = 6$.

Besides, since F and F' lie on a horizontal line, the transverse axis is horizontal,

Using the relation $b^2 = c^2 - a^2 = 9 - 4 = 5$, the equation becomes

$$\frac{(x+1)^2}{4} - \frac{(y-5)^2}{5} = 1 .$$

The graph of the hyperbola is given in Figure 3.27.

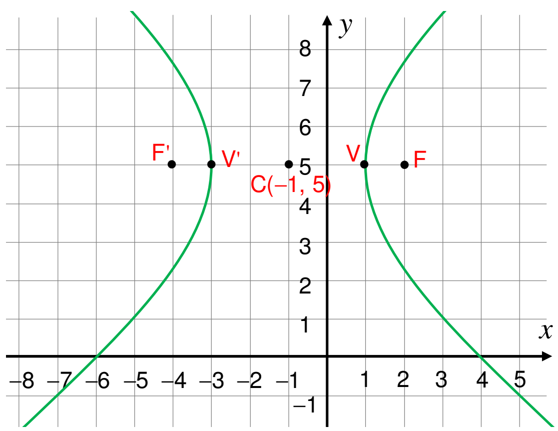
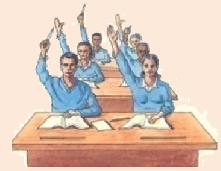


Figure 3.27

ACTIVITY 3.7



Consider the hyperbola with equation

$$\frac{x^2}{9}-\frac{y^2}{16}=1$$

and answer each of the following.

a Draw the graph of the hyperbola with the equation given above.

b Mark the points with coordinates $(\pm 3,\ 0)$ on the x-axis and with coordinates $(0,\ \pm 4)$ on the y-axis.

c Draw a rectangle with sides passing through the points in b above and parallel to the coordinate axes.

d Draw the lines that contain the diagonals of the rectangle in c above.

Asymptotes

If a point P on a curve moves farther and farther away from the origin, and the distance between P and some fixed line tends to zero, then such a line is called **an asymptote to the curve**.

From Activity 3.7 you may have observed that the lines through the diagonals of a rectangle that passes through points with coordinates $(\pm 3,\ 0)$ on the x-axis and $(0,\ \pm 4)$ on the y-axis and parallel to the coordinate axes are asymptotes to the graph of the hyperbola with equation

$$\frac{x^2}{9}-\frac{y^2}{16}=1.$$

Consider the hyperbola with equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

This equation is equivalent to

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=1$$

or

$$\frac{x}{a}-\frac{y}{b}=\frac{ab}{bx+ay}$$

One branch of the hyperbola lies in the first quadrant. If a point P on the hyperbola moves farther and farther away from the origin on this branch of the hyperbola, then x and y become infinite and

$$\frac{ab}{bx+ay}$$

tends to zero. This implies the line

$$\frac{x}{a}-\frac{y}{b}=0 \quad \text{or} \quad y=\frac{b}{a}x$$

is an asymptote to the graph of the hyperbola.

By symmetry, the line

$$\frac{x}{a}+\frac{y}{b}=0 \quad \text{or} \quad y=-\frac{b}{a}x$$

is also an asymptote to the graph of the hyperbola.

If you interchange x and y in the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

the new equation becomes

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

and represents a hyperbola with foci F(0, $-c$) and F'(0, c), vertices V(0, $-a$) and V'(0, a), co-vertices B($-b$, 0) and B′(b, 0), centre C(0, 0), the transverse axis is on y-axis. In this case, the lines $y=\pm\frac{a}{b}x$ are asymptotes to the graph of the hyperbola.

Let C (h, k) be the centre of the hyperbola. Construct a new $x'y'$-coordinate system with origin at (h, k). Then, for any point P on the hyperbola with coordinates (x, y) in the xy -coordinate system and (x', y') in the new $x'y'$-coordinate system,

$$\frac{x'^2}{a^2}-\frac{y'^2}{b^2} = 1$$

Using translation formulae $x' = x - h$ and $y' = y - k$, this reduces to

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

which is the standard equation of a hyperbola with centre at C(h, k) and transverse axis parallel to the x-axis.

Similarly, when the transverse axis is vertical, the standard equation of the hyperbola is given by:

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1, \text{ when C }(0\ ,0)\ \text{ and}$$

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1, \quad \text{when C}(h,k)$$

The following table gives all possible standard forms of equations of hyperbolas.

Equation	Centre	Transverse axis	Asymptotes
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	(0, 0)	horizontal	$y=\pm\frac{b}{a}x$
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	(h, k)	horizontal	$y-k=\left(\pm\frac{b}{a}(x-h)\right)$
$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$	(0, 0)	vertical	$y=\pm\frac{a}{b}x$
$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$	(h, k)	vertical	$y-k=\left(\pm\frac{a}{b}(x-h)\right)$

Example 19 Find asymptotes of the hyperbola, if the foci are F(2, 5) and F'(–4, 5) and the transverse axis is 4 units long.

Solution From Example 18, the equation of the hyperbola is:

$$\frac{(x+1)^2}{4}-\frac{(y-5)^2}{5}=1$$

The asymptotes of the hyperbola are:

$$y-k=\pm\left(\frac{b}{a}(x-h)\right).$$

That is $y-5=\pm\left(\frac{\sqrt{5}}{2}(x+1)\right) \Rightarrow y=\pm\left(\frac{\sqrt{5}}{2}(x+1)\right)+5$

which gives the lines with equations

$$y=\frac{\sqrt{5}}{2}x+\frac{\sqrt{5}+10}{2}, \text{ and } y=-\frac{\sqrt{5}}{2}x+\frac{10-\sqrt{5}}{2}.$$

Example 20 Find the equation of the hyperbola with vertices (1, 2) and (1, – 2), and $b=2$.

Solution The vertices lie on a vertical line. Thus, the equation must have the form

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

The centre is mid way between (1, 2) and (1, –2). So, C(1, 0).

Also $2a = VV' = 4 \Rightarrow a = 2$.

It follows that the equation is $\frac{(y-0)^2}{4} - \frac{(x-1)^2}{4} = 1$

or $\frac{y^2}{4} - \frac{(x-1)^2}{4} = 1$

Example 21 Sketch the hyperbola with equation:

$16y^2 - 9x^2 = 144$.

Draw its asymptotes and give the coordinates of its vertices and foci.

Solution The equation $16y^2 - 9x^2 = 144$ is equivalent to $\frac{16y^2}{144} - \frac{9x^2}{144} = 1$.

Therefore, the equation of the hyperbola is $\frac{y^2}{9} - \frac{x^2}{16} = 1$.

This implies the centre is C(0, 0), $a = 3$, $b = 4$, and the vertices of the hyperbola are V'(0, –3), and V(0, 3).

From the relation $c^2 = a^2 + b^2 = 25$, you get $c = 5$.

Hence the foci are F' $(h, k - c)$ and F $(h, k + c)$, which implies F' (0, –5) and F (0, 5).

Asymptotes of the hyperbola are: $y = \pm\frac{a}{b}x$. That is, $y = \pm\frac{3}{4}x$.

The graph of the hyperbola is given in Figure 3.28.

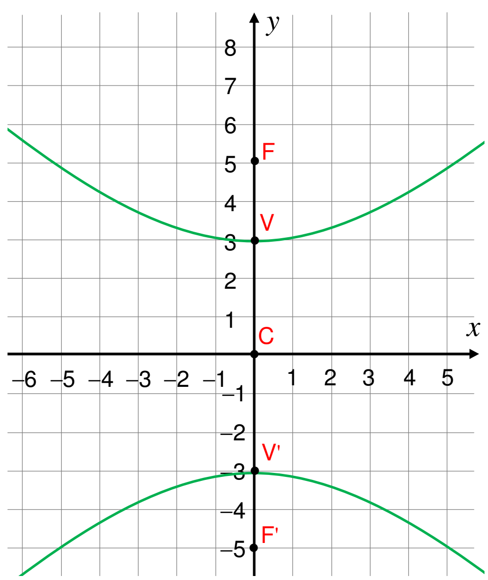


Figure 3.28

Exercise 3.6

1 Find the equation of each hyperbola with the information given below.

a Centre at C(0, 0); $a = 8$, $b = 5$, having horizontal transverse axis.

b Foci at F(10, 0) and F'(–10, 0); $2a = 16$.

c Centre C(–1, 4), $a = 2$, $b = 3$; vertical transverse axis.

d Vertices V(2, 1), V'(–2, 1); $b = 2$.

2 Name the centre, foci, vertices and the equations of the asymptotes of each hyperbola given below. Also sketch their graph.

a $\frac{x^2}{36} - \frac{y^2}{81} = 1$

b $\frac{(x+3)^2}{9} - \frac{(y+6)^2}{36} = 1$

c $\frac{y^2}{25} - \frac{x^2}{16} = 1$

d $\frac{(y-3)^2}{25} - \frac{(x-2)^2}{25} = 1$

3 Write the equation of each hyperbola satisfying the following conditions:

a Centre C(4, –2); focus F(7, –2); vertex V(6, –2)

b Centre C(4, 2); vertex V(4, 5); equation of one asymptote is $4y - 3x = -4$.

c Vertices at V(0, –4), V'(0, 4); foci at F(0, 5), F'(0, –5)

d Vertices at V(–2, 3), V'(6, 3); one focus at F(–4, 3)

e The transverse axis coincides with the x-axis, centre at C(2, 0); lengths of transverse and conjugate axes equal to 8 and 6, respectively.

f The length of the transverse axis is equal to 8; the end points of the conjugate axis are A(5, –5) and B(5, 3).

4 A hyperbola for which $a = b$ is called **equilateral**. Show that a hyperbola is equilateral, if and only if its asymptotes are perpendicular to each other.

Key Terms

angle of inclination	major axis	slope-intercept form
asymptote	minor axis	tangent line
axis	parallel lines	translation formulas
centre	perpendicular lines	transverse axis
conjugate axis	point of tangency	two-point form
directrix	point-slope form	vertex
focal length	radius	*x*-intercept
focus	secant line	*y*-intercept
latus rectum	slope	

Summary

1. The **slope** of a line through (x_1, y_1) and (x_2, y_2) is given by $\frac{y_2 - y_1}{x_2 - x_1}$.
2. **Two point form:** If (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ are given, the line through them has an equation $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$
3. **Point-slope form:** If a point (x_1, y_1) and slope m are given, the equation of the line is $y - y_1 = m(x - x_1)$
4. **Slope-intercept form:** If the slope m and y-intercept b are given, then the equation of the line is $y = mx + b$.
5. Two lines are **parallel** if and only if they have the same angle of inclination.
6. The slope of a **non-vertical** line is tan α, where α is the angle of inclination of the line, with $0 \leq \alpha < 180^o$.
7. The angle β between two non-vertical lines is given by the formula

   $\tan \beta = \frac{m - n}{1 + mn}$, where m and n are the slopes of the lines.
8. Two lines are **perpendicular** if and only if the angle between them is 90^o.
9. If two perpendicular lines are non-vertical, then $mn = -1$, where m and n are their slopes.
10. The **general form of equation of a line** is $Ax + By + C = 0$, where $A \neq 0$ or $B \neq 0$ are fixed real numbers.

11 The distance from the origin to the line $Ax + By + C = 0$ is given by $\dfrac{|C|}{\sqrt{A^2 + B^2}}$

12 The distance from P (h, k) to $Ax + By + C = 0$ is $\dfrac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$

13 If the xy-coordinate system is translated to a new $x'y'$-coordinate system with origin at C(h, k), then the translation formulae are

$$x' = x - h$$

$$y' = y - k$$

14 The **standard form of the equation of a circle** is $(x - h)^2 + (y - k)^2 = r^2$, where (h, k) is the centre and r is the radius.

15 The line that touches a circle at only one point is called a **tangent line** and its equation is $\dfrac{y - y_o}{x - x_o} = \dfrac{-(x_o - h)}{y_o - k}$, where (x_o, y_o) is the **point of tangency** and (h, k) is the **centre** of the circle.

16 The **standard equation of a parabola** is either

$(x - h)^2 = \pm 4p\,(y - k)$ (axis // to the y-axis)

or $(y - k)^2 = \pm 4p\,(x - h)$ (axis // to the x-axis)

17 The **standard equation of an ellipse** is either

$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ (major axis horizontal)

or $\dfrac{(y-k)^2}{a^2} + \dfrac{(x-h)^2}{b^2} = 1$ (major axis vertical)

where $b^2 + c^2 = a^2$

18 The **standard equation of a hyperbola** is either

$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ (transverse axis horizontal)

or $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ (transverse axis vertical)

where $a^2 + b^2 = c^2$

19 The equations of the asymptotes of a hyperbola with a horizontal transverse axis are

$$y - k = \pm \frac{b}{a}(x - h)$$

and those with vertical transverse axis are $y - k = \pm \dfrac{a}{b}(x - h)$

Review Exercises on Unit 3

1 Write each of the following in the general form of equation of a line.

a $y = -3$ **b** $x = 9$ **c** $y = \frac{1}{2}x + 4$

d $y - 3 = 4 - x$ **e** $3x = 7 - 4y$

2 Give the equation of the line that satisfies the given conditions:

a passes through P(–2, 3) and has slope –1.

b passes through P(3, 7) and Q(6, –10).

c parallel to the line with equation $y = 3x - 4$ and passes through A(3, –2).

d perpendicular to the line with equation $6x = 2y - 4$ and y-intercept 4.

3 Find the tangent of the acute angle between the following lines:

a $2x + y - 2 = 0$
$3x + y + 1 = 0$

b $x - 6y + 5 = 0$
$2y - x - 1 = 0$

c $-x - 5y - 2 = 0$
$y - 4x + 7 = 0$

d $x - 6y + 5 = 0$
$2y - x - 1 = 0$

4 Find the distance from the given point to the line whose equation is given.

a P(4, 3); $2x - 3y + 2 = 0$ **b** A(0, 0); $2x - 3y + 2 = 0$

c Q(– 1, 0); $2x - 3y + 2 = 0$ **d** B(– 2, 4); $4y = 3x - 1$

5 Find the distance between the pairs of parallel lines whose equations are given below:

a $2x - 3y + 2 = 0$ and $2x - 3y + 6 = 0$ **b** $4y = 3x - 1$ and $8y = 6x - 7$

6 Write the equation of each circle with the given conditions:

a centre at O(3, –7) and radius 3

b centre at P(3, –7) and tangent to $2x + 3y - 4 = 0$

c end points of its diameter are A(3, –7) and B(4, 3)

7 Find the equation of the tangent line to the circle with equation $(x - 3)^2 + (y - 4)^2 = 20$ at P(1, 0).

8 Find the equation of the parabola with the following conditions.

a focus at F(– 2, 0); directrix $x = 2$

b focus at F(3, 3); vertex at V(3, 2)

c vertex at O(0, 0); axis y-axis; passes through A(−1, 1)

9 For each parabola whose equation is given below, find the focus, vertex, directrix and axis.

a $(x-1)^2 = y+2$ **b** $x^2 = -6y$ **c** $4(x+1) = 2(y+2)^2$

10 Write the equation of each ellipse that satisfies the following conditions.

a The foci are F(3, 0) and F'(−3, 0); vertices V(5, 0) and V'(−5, 0).

b The foci are F(3, 2) and F'(3, −2); the length of the major axis is 8.

c The foci are F(4, 7) and F'(−4, 7); the length of the minor axis is 9.

d The centre is C(6, −2); one focus is F(3, −2) and one vertex is V(10, −2).

11 Find the foci and vertices of each of the ellipses whose equations are given.

a $4x^2 + y^2 = 8$ **b** $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$

12 Give the equation of a hyperbola satisfying the following conditions:

a foci at F(9, 0) and F'(−9, 0); vertices at V(4, 0) and V'(−4, 0).

b foci at F(0, 6) and F'(0, −6); length of transverse axis is 6.

c the foci at F(0, 10) and F' (0, −10); asymptotes $y = \pm 3x$.

13 Find the vertices, foci, eccentricity and asymptotes of each hyperbola whose equation is given and sketch the hyperbola.

a $9x^2 - 16y^2 = 144$ **b** $\frac{(x+3)^2}{25} - \frac{(y+1)^2}{144} = 1$

14 An arch is in the form of a semi-ellipse. It is 50 metres wide at the base and has a height of 20 metres. How wide is the arch at the height of 10 metres above the base?

Hint:- *Take the x-axis along the base and the origin at the midpoint of the base.*

15 An astronaut is to be fired into an elliptical orbit about the earth having a minimum altitude of 800 km and a maximum altitude of 5400 km. Find the equation of the curve followed by the astronaut. Consider the radius of the earth to be 6400 km.

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Unit 4

MATHEMATICAL REASONING

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about mathematical logic.*
- *know methods and procedures in combining and determining the validity of statements.*
- *know basic facts about argument and validity.*

Main Contents

4.1 LOGIC

4.2 ARGUMENTS AND VALIDITY

Key terms

Summary

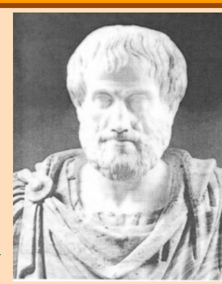
Review Exercises

INTRODUCTION

Mathematical Reasoning is a tool for organizing evidence in a systematic way through mathematical logic. In this unit, you will study mathematical logic, the systematic study of the art of reasoning. In some ways, mathematics can be thought of as a theory of logic. Logic has a wide range of applications, particularly in judging the correctness of a chain of reasoning, as in mathematical proofs.

In the first sub-unit, Logic, you will study the following: statements and open statements, fundamental logical connectives (or logical operators), compound propositions, properties and laws of logical connectives, contradiction and tautology, converse, contrapositive and quantifiers. In the second sub-unit, you will study Arguments, Validity, and rules of inferences.

HISTORICAL NOTE



Aristotle (384 – 322 B.C.)

Aristotle was one of the greatest philosophers of ancient Greece. After studying for twenty years in Plato's Academy, he became tutor to Alexander the Great. Later, he founded his own school, the Lyceum, where he contributed to nearly every field of human knowledge. After Aristotle's death, his treatises on reasoning were grouped together and came to be known as the Organon.

The word "logic" did not acquire its modern meaning until the second century AD, but the subject matter of logic was determined by the content of the Organon.

OPENING PROBLEM

Do you think that the following arguments are acceptable?

Wages will increase only if there is inflation. If there is inflation, then the cost of living will increase. Wages will increase. Therefore, the cost of living will increase.

Confused! Don’t worry! Your study of logic will help you to decide whether or not this given argument is acceptable.

4.1 LOGIC

In this sub-unit, you will learn mathematical logic at its elementary level, known as propositional logic. Propositional logic is the study of assertive or declarative sentences which can be said to be either True denoted by **T** or False denoted by **F**, but not both. The value T or F that is assigned to a sentence is called the **truth value** of the sentence.

4.1.1 "Statement" and "Open Statement"

We begin this subtopic by identifying whether a given sentence can be said to be True, False or neither. We define those sentences which can be said to be True or False, but not both, as **statements** or **propositions**. The following Group Work should lead to the definition.

Group Work 4.1

Discuss the following issues in groups and justify your answers.

1 What is a sentence?

2 Identify whether the following sentences can be said to be True, False or neither and give your reasons.

- **a** Man is mortal.
- **b** Welcome.
- **c** $2 + 5 = 9$
- **d** $4 + 5 = 9$
- **e** God bless you.
- **f** It is impossible to get medicine for HIV/AIDS.
- **g** You can get a good grade in mathematics.
- **h** $x + 6 = 8$
- **i** King Abba Jifar weighed 60 kg when he was 30 years old.
- **j** $x + 3 < 10$
- **k** ____ is a town in Ethiopia.
- **l** x is less than y.

From the above group work, you may have identified the following:

- ✓ Sentences which can be said to be true or false (but not both).
- ✓ Sentences with one or more variables or blank spaces.
- ✓ Sentences which express hopes or opinions, (as opposed to facts).

Definition 4.1

i A sentence which can be said to be true or false (but not both) is said to be a **proposition**(or **statement**).

ii A sentence with one or more variables which becomes a statement on replacing the variable or variables by an individual or individuals is called an **open proposition** (**or open statement**).

iii The words True and False, denoted by T and F respectively, are called **truth values**.

Example 1 From Group Work 4.1 above, you see that

i **a** Man is mortal. **c** $2 + 5 = 9$ **d** $4 + 5 = 9$

i King Abba Jifar weighed 60 kg when he was 30 years old,

are all propositions.

ii **h** $x + 6 = 8$ **j** $x + 3 < 10$ **l** x is less than y.

k ____ is a town in Ethiopia, are all open propositions.

iii **b** Welcome. **f** It is impossible to get medicine for HIV/AIDS.

g You can get a good grade in mathematics. **e** God bless you,

are all neither propositions nor open propositions.

Exercise 4.1

Identify each of the following as a proposition, an open proposition or neither.

a On his 35^{th} birthday, Emperor Tewodros invited 1000 people for dinner.

b Sudan is a country in Africa.

c If x is any real number, then $x^2 - 1 = (x - 1)(x + 1)$.

d You are a good student.

e A square of an even number is even.

f Ambo is a town in Oromiya.

g $8^{90} > 9^{80}$

h God have mercy on my soul!

i x is less than 9.

j _______ is the study of plants.

k For a real number x, $x^2 + 1 < 0$.

l No woman should die while giving birth.

m Laws and orders are dynamic.

n Every child has the right to be free of corporal punishment.

4.1.2 Fundamental Logical Connectives (Operators)

Given two or more propositions, you can use connectives to join the sentences. The fundamental connectives in logic are: **or, and, if … then, if and only if** and **not**. Under this subtopic, you learn how to form a statement which consists of two or more component propositions connected by logical connectives or logical operators. In doing this, you also learn the rules that govern us when communicating through logic. You will begin with the following Activity.

ACTIVITY 4.1

Consider the following propositions.

Water is a natural resource. (True)

Plants do not need water to grow. (False)

Work is an instrument for national development. (True)

Everyone does not have the right to hold opinions without interference. (False)

Determine the truth value of each of the following:

a Water is **not** a natural resource.

b Plants need water to grow.

c Water is a natural resource **and** plants need water to grow.

d Water is a natural resource **or** plants need water to grow.

e **If** water is a natural resource, **then** plants need water to grow.

f Water is a natural resource, **if and only if** plants need water to grow.

g Water is **not** a natural resource **or** work is an instrument for national development.

h Work is an instrument for national development, **if and only if** everyone does not have the right to hold opinions without interference.

i **If** water is **not** a natural resource, **then** plants need water to grow.

j **If** everyone has no right to hold opinions without interference, **then** work is an instrument for national development.

To find the truth-value of a statement which is combined by using connectives, you need rules which give the truth value of the compound statement. You also need symbols for connectives and notations for propositions. You usually represent propositions by small letters such as p, q, r, s, t, and so on. Now, let p represent one proposition and q represent another proposition.

Connective	Name of the connective	Symbol	How to write	How to read
not	negation	$\neg$	$\neg p$	The negation of p
and	conjunction	$\wedge$	$p \wedge q$	p and q
or	disjunction	$\vee$	$p \vee q$	p or q
If…, then…	implication	$\Rightarrow$	$p \Rightarrow q$	p implies q
If and only if	Bi-implication	$\Leftrightarrow$	$p \Leftrightarrow q$	p if and only if q

Example 2 Let p represent the proposition: Water is a natural resource.

Let q represent the proposition: Plants need water to grow. Then,

a $\neg p$ represents: Water is not a natural resource.

b $p \wedge q$ represents: Water is a natural resource and plants need water to grow.

c $p \vee q$ represents: Water is a natural resource or plants need water to grow.

d $p \Rightarrow q$ represents: If water is a natural resource, then plants need water to grow.

e $p \Leftrightarrow q$ represents: Water is a natural resource, if and only if plants need water to grow.

Now we will see to the rules that govern us in communicating through logic by using **truth tables** for each of the logical operators.

Rule 1 Rule for Negation ("¬")

Let p be a proposition.

Then as shown from the table below, its negation is represented by $\neg p$.

Note:

$\neg p$ is true, if and only if p is false.

This is best explained by the following table called the truth table for negation.

p	$\neg p$
T	F
F	T

Example 3 p: Work is an instrument for national development. (True)

$\neg p$: Work is not an instrument for national development. (False)

q: Nairobi is the capital city of Ethiopia. (False)

$\neg q$: Nairobi is not the capital city of Ethiopia. (True)

Note:

The word "not" denoted by "¬" is applied to a single statement and does not connect two statements, as a result of this, the name logical operator is appropriate for it.

Rule 2 Rule for Conjunction ("$\wedge$")

When two propositions p and q are joined with the connective **"and"** (denoted by $p \wedge q$), the proposition formed is a logical conjunction. In this case p and q are called **the components of the conjunction**.

$p \wedge q$ is true, if and only if both p and q are true.

To determine the truth value of $p \wedge q$, we have to know the truth value of the components p and q.

The possibilities are as follows:

p true and q true

p true and q false

p false and q true

p false and q false.

This is illustrated by the following truth table.

The truth table for conjunction is given as:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 4 Consider the following propositions:

p: Work is an instrument for national development. (True)

q: Nairobi is the capital city of Ethiopia. (False)

r: $2 < 3$ (True)

a $p \wedge q$: Work is an instrument for national development and Nairobi is the capital city of Ethiopia. (False)

b $p \wedge \neg q$: Work is an instrument for national development and Nairobi is not the capital city of Ethiopia. (True)

c $p \wedge r$: Work is an instrument for national development and $2 < 3$. (True)

Rule 3 Rule for Disjunction ("$\vee$")

When two propositions p and q are joined with the connective **"or"** (denoted by $p \vee q$), the proposition formed is a logical disjunction.

$p \vee q$ is false, if and only if both p and q are false.

To determine the truth value of $p \vee q$, we have to know the truth value of the components p and q. As mentioned earlier, if we have two propositions to be combined, there are four possibilities of combinations of the truth values of component propositions.

The truth table for disjunction is given as:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 5 Consider the following propositions

p: Work is an instrument for national development. (True)

q: Nairobi is the capital city of Ethiopia. (False)

r: $2 < 3$ (True)

a $p \vee q$: Work is an instrument for national development or Nairobi is the capital city of Ethiopia. (True)

b $q \vee r$: Nairobi is the capital city of Ethiopia or $2 < 3$. (True)

c $q \vee \neg r$: Nairobi is the capital city of Ethiopia or $2 \geq 3$. (False)

Rule 4 Rule for Implication ("$\Rightarrow$")

When two propositions p and q are joined with the connective **"implies"** (denoted by $p \Rightarrow q$) the proposition formed is a logical implication.

$p \Rightarrow q$ is false, if and only if p is true and q is false.

This is illustrated by the truth table for implication which is given as follows:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 6 Consider the following propositions:

p: Work is an instrument for national development. (True)

q: Nairobi is the capital city of Ethiopia. (False)

r: $2 < 3$ (True)

a $p \Rightarrow q$: If work is an instrument for national development, then Nairobi is the capital city of Ethiopia. (False)

b $q \Rightarrow r$: If Nairobi is the capital city of Ethiopia, then $2 < 3$. (True)

c $q \Rightarrow \neg r$: If Nairobi is the capital city of Ethiopia, then $2 \geq 3$. (True)

d $\neg q \Rightarrow r$: If Nairobi is not the capital city of Ethiopia, then $2 < 3$. (True)

Rule 5 Rule for Bi-implication ("if and only if")

When two propositions p and q are joined with the connective **"bi-implication"** (denoted by $p \Leftrightarrow q$) the proposition formed is a logical bi-implication.

$p \Leftrightarrow q$ is false, if and only if p and q have different truth values.

This is illustrated by the truth table for bi–implication which is given as follows:

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 7 Consider the following propositions

p: Work is an instrument for national development. (True)

q: Nairobi is the capital city of Ethiopia. (False)

r: $2 < 3$ (True)

a $p \Leftrightarrow q$: Work is an instrument for national development, if and only if Nairobi is the capital city of Ethiopia. (False)

b $q \Leftrightarrow r$: Nairobi is the capital city of Ethiopia, if and only if $2 < 3$. (False)

c $q \Leftrightarrow \neg r$: Nairobi is the capital city of Ethiopia, if and only if $2 \geq 3$. (True)

d $\neg q \Leftrightarrow r$: Nairobi is not the capital city of Ethiopia, if and only if $2 < 3$. (True)

Exercise 4.2

Given that p: Man is mortal.

q: Botany is the study of plants.

r: 6 is a prime number.

Determine the truth values of each of the following.

a $p \wedge q$	**d** $\neg p \vee q$	**g** $\neg p \wedge \neg q$
b $(p \wedge q) \Rightarrow r$	**e** $\neg (p \vee q)$	**h** $\neg p \vee \neg q$
c $(p \wedge q) \Leftrightarrow \neg r$	**f** $\neg (p \wedge q)$	**i** $p \Leftrightarrow q$

4.1.3 Compound Statements

So far, you have defined statements and logical connectives (or logical operators) and you have seen the rules that go with the logical connectives. Now you are going to give a name for statements formed from two or more component propositions by using logical operators. Each "sentence" a - i in Exercise 4.2 is a statement formed by using one or more connectives.

Definition 4.2

A statement formed by joining two or more statements by a connective (or connectives) is called a **compound statement**.

Example 8 Consider the following statements:

p: 3 divides 81. (True)

q: Khartoum is the capital city of Kenya. (False)

r: A square of an even number is even. (True)

s: $\frac{22}{7}$ is an irrational number. (False)

Determine the truth value of each of the following:

a $(p \wedge q) \Rightarrow (r \vee s)$ **b** $(\neg p \vee q) \wedge (r \wedge s)$

c $(p \wedge r) \Leftrightarrow (q \wedge s)$ **d** $(r \vee s) \wedge (p \wedge \neg q)$

Solution:

a $p \wedge q$ has truth value F, $r \vee s$ has truth value T, thus $(p \wedge q) \Rightarrow (r \vee s)$ has truth value T.

b $(\neg p \vee q)$ has truth value F, and hence $(\neg p \vee q) \wedge (r \vee s)$ has truth value F.

c $(p \wedge r)$ has truth value T, $(q \wedge s)$ has truth value F and hence

$(p \wedge r) \Leftrightarrow (q \wedge s)$ has truth value F.

d $(r \vee s)$ has truth value T and $(p \wedge \neg q)$ has truth value T, hence

$(r \vee s) \wedge (p \wedge \neg q)$ has truth value T.

Example 9 Let p, q, r have truth values T, F, T, respectively. Determine the truth value of each of the following.

a $\neg p \vee q$ **b** $\neg p \wedge \neg q$ **c** $(p \vee q) \Rightarrow r$

Solution:

a Since p has truth value T, then $\neg p$ has truth value F.

$\neg p$ has truth value F and q also has truth value F.

Thus $\neg p \vee q$ has truth value F by the rule of logical disjunction.

b From (a) $\neg p$ has truth value F.

q has truth value F, and hence $\neg q$ has truth value T.

Thus $\neg p \wedge \neg q$ has truth value F by the rule of conjunction.

c Since p has truth value T and q has truth value F.

$p \vee q$ has truth value T by the rule of disjunction.

Since r has truth value T, $(p \vee q) \Rightarrow r$ has truth value T by the rule of implication.

Example 10 Let p and q be any two propositions. Construct one truth table for each of the following pairs of compound proposition and compare their truth values.

a $p \Rightarrow q$, $\neg p \vee q$ **c** $p \Rightarrow q$, $\neg q \Rightarrow \neg p$

b $\neg (p \vee q)$, $\neg p \wedge \neg q$ **d** $p \Rightarrow q$, $q \Rightarrow p$

Solution We construct the truth table as follows:

a

p	q	$\neg p$	$p \Rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Observe that both $p \Rightarrow q$ and $\neg p \vee q$ have the same truth values.

b

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg (p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Observe that both $\neg (p \vee q)$ and $\neg p \wedge \neg q$ have the same truth values.

c

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Observe that both $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$ have the same truth values.

d

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Observe that $p \Rightarrow q$ and $q \Rightarrow p$ do not have the same truth table. As you have seen from Example 10, some compound propositions have the same truth values for each assignment of the truth values of component propositions. Such pairs of compound propositions are called **equivalent propositions**. We use the symbol "≡" in-between the two propositions to mean they are equivalent.

Thus, from observation of the tables for Example 10, we have:

a $p \Rightarrow q \equiv \neg p \vee q$

b $\neg(p \vee q) \equiv \neg p \wedge \neg q$

c $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

d $p \Rightarrow q$ and $q \Rightarrow p$ are not equivalent.

Exercise 4.3

1 Let p, q, r have truth values T, F, T respectively, then determine the truth value of each of the following:

a $\neg(p \vee q)$ **b** $(\neg p \vee q) \Rightarrow r$ **c** $(p \wedge q) \Rightarrow r$

d $(p \vee q) \Rightarrow \neg r$ **e** $(p \wedge q) \Leftrightarrow r$

2 Given p: The sun rises due East.

q: 5 is less than 2.

r: Pigeons are birds.

s: Laws and orders are dynamic.

t: Lake Tana is found in Ethiopia.

Express each of the following compound propositions in good English and determine the truth value of each.

a $p \wedge r$ **b** $p \vee r$ **c** $(p \wedge r) \Rightarrow q$

d $(p \wedge \neg r) \Leftrightarrow \neg q$ **e** $p \Rightarrow (q \vee r)$ **f** $p \Leftrightarrow (q \wedge r)$

g $s \Rightarrow t$ **h** $s \Leftrightarrow t$ **i** $s \wedge t$

3 Construct the truth table for each of the following compound statements.

a $p \Rightarrow (p \Rightarrow q)$ **b** $p \Rightarrow \neg (p \wedge r)$

c $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$ **d** $(p \wedge q) \Leftrightarrow (p \vee q)$

4 Suppose the truth value of $p \Rightarrow q$ is T.

What can be said about the truth value of $(p \wedge q) \Leftrightarrow (p \vee q)$?

5 Suppose the truth value of $p \Leftrightarrow q$ is T.

What can be said about the truth values of

a $p \Leftrightarrow \neg q$? **b** $\neg p \Leftrightarrow q$? **c** $\neg p \Leftrightarrow \neg q$?

4.1.4 Properties and Laws of Logical Connectives

Under this subtopic, you are going to see some of the properties of logical connectives and discuss commutative, associative and distributive properties in the sense of equivalence and also see other properties known as De Morgan's Laws. The following Activity will help you to have more understanding of these properties.

ACTIVITY 4.2

Construct truth tables for each of the following pairs of compound propositions and check whether the given pairs are equivalent or not.

a $p \wedge q, q \wedge p$ **b** $p \vee q, q \vee p$

c $p \wedge (q \wedge r), (p \wedge q) \wedge r$ **d** $p \vee (q \vee r), (p \wedge q) \vee r$

e $p \wedge (q \vee r), (p \wedge q) \vee (p \vee r)$ **f** $p \vee (q \wedge r), (p \vee q) \wedge (p \vee r)$

g $\neg p \vee \neg q$, $\neg (p \wedge q)$ **h** $\neg p \wedge \neg q$, $\neg (p \vee q)$

From the above activity, you should have observed that the following properties hold true.

1 Conjunction is **commutative**; that means for any propositions p and q, we have

$$p \wedge q \equiv q \wedge p$$

2 Disjunction is **commutative**; that means for any propositions p and q, we have

$$p \vee q \equiv q \vee p$$

3 Conjunction is **associative**; that means for any propositions p, q and r, we have

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

4 Disjunction is **associative**; that means for any propositions p, q and r, we have

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

5 Conjunction is **distributive over disjunction**; that means for any propositions p, q and r, we have

$$(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

6 Disjunction is **distributive over conjunction**; that means for any propositions p, q and r, we have

$$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

7 You have also seen that

$$\neg p \vee \neg q \equiv \neg (p \wedge q)$$

$$\neg p \wedge \neg q \equiv \neg (p \vee q)$$

These two properties are called **De Morgan's Laws**.

4.1.5 Contradiction and Tautology

Begin this subsection by doing the following Group Work.

Group Work 4.2

Complete the truth table for each of the following compound propositions in the following tables and discuss the results.

a $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$ **b** $(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$

c $(p \vee q) \Leftrightarrow (p \vee \neg q)$

a

p	q	$\neg p$	$p \Rightarrow q$	$\neg p \vee q$	$(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$
T	T				
T	F				
F	T				
F	F				

i From the above truth table, what did you observe about the truth values of $(p \Rightarrow q) \Leftrightarrow (\neg p \vee q)$?

ii Is the last column always true?

iii Is the last column always false?

b

p	q	$\neg q$	$p \Rightarrow q$	$p \wedge \neg q$	$(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T				
T	F				
F	T				
F	F				

i From the above truth table, what did you observe about the truth values of $(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$?

ii Is the last column always true?

iii Is the last column always false?

c

p	q	$\neg q$	$p \vee q$	$p \vee \neg q$	$(p \vee q) \Leftrightarrow (p \vee \neg q)$
T	T				
T	F				
F	T				
F	F				

i From the above truth table what did you observe about the truth values of $(p \vee q) \Leftrightarrow (p \vee \neg q)$?

ii Is the last column always true?

iii Is the last column always false?

The following definition refers to the observations made in Group Work 4.2 above:

Definition 4.3

a A compound proposition is a **tautology**, if and only if for every assignment of truth values to the component propositions occurring in it, the compound proposition always has truth value T.

b A compound proposition is a **contradiction**, if and only if for every assignment of truth values to the component propositions occurring in it, the compound proposition always has truth value F.

Note that in the above group work (c) is neither a tautology nor a contradiction.

Exercise 4.4

Determine whether each of the following compound propositions is a tautology, a contradiction or neither.

a $(p \wedge q) \Leftrightarrow (q \wedge p)$

b $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$

c $[p \wedge (q \wedge r)] \Leftrightarrow [(p \wedge q) \wedge r]$

d $[p \vee (q \vee r)] \Leftrightarrow [\neg (p \wedge q) \wedge \neg r]$

e $[p \wedge (q \vee r)] \Leftrightarrow [\neg (p \wedge q) \vee \neg (p \vee r)]$

f $[\neg p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge (p \vee r)]$

g $(\neg p \vee \neg q) \Leftrightarrow (p \wedge q)$

h $(\neg p \wedge \neg q) \Rightarrow \neg(p \vee q)$

4.1.6 Converse and Contrapositive

Mathematical statements (or assertions) are usually given in the form of a conditional statement $p \Rightarrow q$. You will now examine such conditional statements.

ACTIVITY 4.3

Consider the following statements.

p: A child has the right to be free from corporal punishment.

q: The sun rises due north.

Write the following in good English.

a $p \Rightarrow q$ **b** $q \Rightarrow p$ **c** $\neg q \Rightarrow \neg p$

You may recall from Example 10 that $p \Longrightarrow q \equiv \neg q \Longrightarrow \neg p$ and $p \Longrightarrow q \not\equiv q \Longrightarrow p$.

Now you will learn the name of these relations in the following definition.

Definition 4.4

Given a conditional statement $p \Rightarrow q$

a $q \Longrightarrow p$ is called the converse of $p \Rightarrow q$

b $\neg q \Rightarrow \neg p$ is called the contrapositve of $p \Rightarrow q$.

c In $p \Rightarrow q$, p is said to be a hypothesis or sufficient condition for q; q is said to be the conclusion or necessary condition for p.

Example 11 Consider the following:

p: A quadrilateral is a square.

q: A quadrilateral is a rectangle.

Write the following conditional statements in good English and determine the truth values of each.

a $p \Rightarrow q$ **b** $q \Rightarrow p$ **c** $\neg q \Rightarrow \neg p$

Solution:

a If a quadrilateral is a square, then it is a rectangle. (True)

b If a quadrilateral is a rectangle, then it is a square. (False)

c If a quadrilateral is not a rectangle, then it is not a square. (True)

Often mathematical statements (or theorems) are given in the form of conditional statements. To prove such statements you can assume that the hypothesis is true and show that the conclusion is also true. But if this approach becomes difficult, you might use a kind of proof called "**proof by contrapositive**". You can appreciate this method of proof if you compare the conditional statement

$p \Rightarrow q$ with its contrapositive $\neg q \Rightarrow \neg p$.

The following example illustrates this.

Example 12 Prove the following assertions.

a If a natural number k is odd, then its square is also odd.

b If a natural number k is even, then its square is also even.

c If k is a natural number and k^2 is even, then k is even.

Proof:

a First you identify the hypothesis and the conclusion.

Hypothesis p: k is an odd natural number.

Conclusion q: k^2 is odd.

The statement is in the form of $p \Rightarrow q$.

Now k is odd implies that $k = 2n - 1$, for some natural number n.

$\Rightarrow k^2 = (2n - 1)^2 = 4n^2 - 4n + 1 = 2\,(2n^2 - 2n + 1) - 1.$

$\Rightarrow k^2 = 2m - 1$, where $m = 2n^2 - 2n + 1$ is a natural number.

$\Rightarrow k^2$ is odd.

Therefore, the assertion is proved.

b Hypothesis p: k is an even natural number.

Conclusion q: k^2 is even.

The statement is in the form of $p \Rightarrow q$.

Now k is even implies that $k = 2n$ for same natural number n.

$\Rightarrow k^2 = (2n)^2 = 4n^2 = 2\,(2n^2)$

$\Rightarrow k^2 = 2m,$ where $m = 2n^2$ is also a natural number.

$\Rightarrow k^2$ is even.

Therefore, the assertion is proved.

c Hypothesis p: k is natural number and k^2 is even.

Conclusion q: k is even.

The statement is in the form of $p \Rightarrow q$.

You may use proof by contrapositive.

Assume that k is not even; that is $\neg\, q$ is true.

k is not even implies that k is odd.

$\Rightarrow k^2$ is odd, by (a)

$\Rightarrow \neg\, p$ is true

$\Rightarrow p$ is false

This contradicts the given hypothesis and hence the assumption that k is not even is false.

Therefore, k must be even.

Exercise 4.5

1 Construct the truth table of the following and compare the truth values of each:

a $\neg(p \Rightarrow q)$ **b** $\neg p \Rightarrow \neg q$ **c** $p \wedge \neg q$

Which one is equivalent to $\neg(p \Rightarrow q)$?

2 For each of the following conditional statements, give the converse and contrapositive.

a If $2 > 3$, then 6 is prime.

b If Ethiopia is in Asia, then Sudan is in Africa.

c If Ethiopia were in Europe, then life would be simpler.

3 Prove that, if k is a natural number such that k^2 is odd, then k is odd.

4.1.7 Quantifiers

Open statements can be converted into statements by replacing the variable (s) by an individual entity. In this section, you are going to see how open statements can be converted into statements by using quantifiers.

ACTIVITY 4.4

Consider the following open statements.

P(x): $x + 5 = 7$; where x is a natural number.

Q(x): $x^2 \geq 0$; where x is a real number.

Can you determine the truth value of the following?

a There is a natural number x such that $x + 5 = 7$.

b For all natural numbers x, $x + 5 = 7$.

c There is a real number x for which $x^2 \geq 0$.

d For every real number x, $x^2 \geq 0$.

You use the symbol $\exists$ for the phrase **"there is"** or "**there exists**" and call it an **existential** quantifier; you use the symbol $\forall$ for the phrase **"for all"** or **"for every"** or **"for each"** and call it a **universal quantifier.**

Thus, you can rewrite the above statements a to d as follows using the symbols and read them as follows:

a $(\exists x)\, P(x) \equiv$ there is **some natural number** which satisfies property P.

Or there is **at least one natural number** which satisfies property P.

b $(\forall x)\, P(x) \equiv$ **all natural numbers** satisfy property P.

Or **every natural number** satisfies property P.

Or **each natural number** satisfies property P.

c $(\exists x)\, Q(x) \equiv$ there is **some real number** which satisfies property Q.

d $(\forall x)\, Q(x) \equiv$ **every real** number satisfies property Q.

Actually, when we attach quantifiers to open propositions, they are no longer open propositions. For example, $(\exists x)\, P(x)$ is **true,** if there is some individual in the given universe which satisfies property P; it becomes **false** if there is no such individual in the universe which satisfies property P. Likewise, $(\forall x)\, P(x)$ is **true,** if all individuals in the universe satisfy property P; it becomes **false** if there is at least one individual in the universe which does not satisfy property P. That means, $(\exists x)\, P(x)$ and $(\forall x)\, P(x)$ have got truth values and they become propositions.

Example 13 Let S = {2, 4, 5, 6, 8, 10} and $P(x)$: x is a multiple of 2 where $x \in$ S.

Determine the truth values of the following.

a $(\exists x)\, P(x)$ **b** $(\forall x)\, P(x)$

Solution:

a $(\exists x)\, P(x)$ is true, since 8 satisfies property P. [Are there other elements of S which satisfy property P?]

b $(\forall x)\, P(x)$ is false, since 5 does not satisfy property P.

Exercise 4.6

Determine the truth value of each of the following assuming that the universe is the set of real numbers.

a $(\exists x)\ (4x - 3 = -2x + 1)$ **b** $(\exists x)\ (x^2 + x + 1 = 0)$

c $(\exists x)\ (x^2 + x + 1 > 0)$ **d** $(\exists x)\ (x^2 + x + 1 < 0)$

e $(\forall x)\ (x^2 > 0)$ **f** $(\forall x)\ (x^2 + x + 1 \neq 0)$

g $(\forall x)\ (4x - 3 = -2x + 1)$

Relations between quantifiers

Given a proposition, it is obvious that its negation is also a proposition. This leads to the question:

What is the form of the negation of $(\exists x)P(x)$ and the form of the negation of $(\forall x)P(x)$?

Group Work 4.3

Let $P(x)$ be an open statement.

Discuss the following: When do you say that

1 $(\exists x)\ P\ (x)$ is true? **2** $(\forall x)\ P(x)$ is true?

3 $(\exists x)\ P\ (x)$ is false? **4** $(\forall x)\ P(x)$ is false?

From the above Group Work you should be able to summarize the following:

The proposition $(\forall x)P(x)$ will be false only if we can find an individual "a" such that $P(a)$ is false, which means $(\exists x)\neg P(x)$ is true. If we succeed in getting such an individual "a", then $(\forall x)P(x)$ is false. Therefore, the negation of $(\forall x)P(x)$ becomes $(\exists x)\neg P(x)$. In symbols, this is

$$\neg(\forall x)P(x) \equiv (\exists x)\neg P(x)$$

To find the symbolic form of the negation of $(\exists x)P(x)$, proceed as follows: $(\exists x)P(x)$ is false if there is no individual "a" for which $P(a)$ is true.

Thus for every x, $P(x)$ is false, which means for every x, the negation of $P(x)$ is true. Therefore, the negation of $(\exists x)P(x)$ becomes $(\forall x)\neg P(x)$. In symbols, this is

$$\neg(\exists x)P(x) \equiv (\forall x)\neg P(x)$$

Example 14 Give the negation of each of the following statements and determine the truth values of each assuming that the universe is the set of all real numbers.

a $(\exists x)\ (x^2 < 0)$ **b** $(\forall x)\ (2x - 1 = 0)$

Solution

a $\neg\ (\exists x)\ (x^2 < 0)\ \equiv (\forall x)\ \neg\ (x^2 < 0) \equiv (\forall x)\ (x^2 \geq 0)$

$(\exists x)\ (x^2 < 0)$ is false; and $(\forall\ x)\ (x^2 \geq 0)$ is true.

b $\neg\ (\forall x)\ (2x - 1 = 0) \equiv (\exists x)\neg\ (2x - 1 = 0) \equiv (\exists x)\ (2x - 1 \neq 0)$

$(\forall x)\ (2x - 1 = 0)$ is false; and $(\exists x)\ (2x - 1 \neq 0)$ is true.

Exercise 4.7

1 Give the negation of each of the following statements and determine the truth values for each, assuming that the universe is the set of real numbers.

a $(\exists x)\ (4x - 3 = -2x + 1)$ **b** $(\exists x)\ (x^2 + 1 = 0)$

c $(\forall x)\ (x^2 + 1 > 0)$ **d** $(\forall x)\ (x^2 < 0)$

e $(\exists x)\ (x^2 + x + 1 = 0)$

2 Let U = {1, 2, 3, 4, 5} be a given universe.

$P(x)$: x is an even number

$H(x)$: x is a multiple of 2

$R(x)$: x is an odd prime number

$Q(x)$: $x \leq 5$.

Determine the truth value of each of the following

a $(\exists x)\ P(x)$ **b** $(\exists x)\ (P(x) \wedge \mathrm{H}(x))$

c $(\exists x)\ (P(x) \Rightarrow \mathrm{H}(x))$ **d** $(\forall x)\ (R(x) \Rightarrow \mathrm{P}(x))$

e $\neg\left[(\forall x)\ (P(x) \Rightarrow \mathrm{H}(x))\right]$ **f** $(\forall x)\ Q(x)$ **g** $(\exists x)\ R(x)$

Quantifiers occurring in combinations

Under this subtopic, you are going to see how to convert an open statement involving two variables into a statement. It involves the use of two quantifiers together or one of the quantifiers twice. To begin with the following Activity may help you.

ACTIVITY 4.5

Answer the following questions:

1 For each natural number, can you find a natural number that is greater than it?

2 For each natural number, can you find a natural number that is less than it?

3 For each integer, can you find an integer that is less than it?

4 Given an integer x, can you find an integer y such that $x + y = 0$?

5 Is there an integer x for every integer y such that:

a $x + y = y$? **b** $x + y = x$?

Observe that each question in the above Activity involves two variables and hence you need either one quantifier twice or the two quantifiers together to covert the open statements into statements.

Suppose you have an open proposition involving two variables, say

$P(x, y): \ x + y = 5$, where x and y are natural numbers.

This open proposition can be changed to a proposition either by replacing both variables by certain numbers explicitly or by using quantifiers. To use quantifiers, either you have to use one of the quantifier twice or both quantifiers in combination. So it is important to know how to read and write such quantifiers. The following will give you plenty of practice!

$(\exists x)(\exists y)P(x, y) \equiv$ There is some x and some y so that property P is satisfied.

This statement is true if one can succeed in finding one individual x and one individual y which satisfy property P.

$(\exists x)(\forall y)P(x, y) \equiv$ There is some x so that property P is satisfied for every y.

$\equiv$ There is some x which stands for all y so that property P is satisfied.

This statement is true, if one can succeed in finding one individual x for which property P is satisfied by every value of y.

$(\forall x)(\exists y)P(x, y) \equiv$ For every x there is some y so that property P is satisfied.

$\equiv$ Given x we can find y so that property P is satisfied.

This statement is true if one can succeed in finding one individual y corresponding to a given x so that property P is satisfied.

$(\forall x)(\forall y)P(x, y) \equiv$ For every x and every y property P is satisfied.

This statement is false if one can succeed in finding an individual x or an individual y which does not satisfy property P.

Thus, if we apply this for the open statement:

$P(x, y): \ x + y = 5$, where x and y are natural numbers, we have.

$(\exists x)(\exists y)P(x, y)$, has truth value T. (You can take $x = 1$ and $y = 4$)

$(\exists x)(\forall y)P(x, y)$, has truth value F.

$(\forall x)(\exists y)P(x, y)$, has truth value F, since if x is given to be 6, for example, we cannot find a natural number y so that $6 + y = 5$.

$(\forall x)(\forall y)P(x, y)$, has truth value F.

But if we change the universe from natural numbers to integers as:

$P(x, y): \ x + y = 5$, where x and y are integers, then

$(\exists x)(\exists y)P(x, y)$, has truth value T.

$(\exists x)(\forall y)P(x, y)$, has truth value F.

$(\forall x)(\exists y)P(x, y)$, has truth value T, since given x we can take $y = 5 - x$ which is also an integer, and satisfies P.

$(\forall x)(\forall y)P(x, y)$, has truth value F.

Exercise 4.8

1. Given $Q(x, y)$: $x = y$ and $H(x, y)$: $x > y$, determine the truth value of each of the following assuming the universe to be the set of natural numbers.

   a $(\exists x)(\forall y)Q(x, y)$ **b** $(\forall x)(\forall y)H(x, y)$ **c** $(\forall x)(\forall y)Q(x, y)$

   d $(\forall y)(\forall x)Q(x, y)$ **e** $(\exists x)(\forall y)H(x, y)$ **f** $(\exists x)(\exists y)H(x, y)$

   g $(\forall x)(\exists y)H(x, y)$

2. Given $P(x, y)$: $y = x + 5$; $Q(x, y)$: $x = y$ and $H(x, y)$: $x > y$; determine the truth value of each of the following, if the universe is the set of real numbers.

   a $(\exists x)(\exists y)P(x, y)$ **b** $(\exists x)(\forall y)P(x, y)$ **c** $(\forall x)(\forall y)P(x, y)$

   d $(\forall x)(\exists y)P(x, y)$ **e** $(\exists x)(\forall y)Q(x, y)$ **f** $(\forall x)(\forall y)H(x, y)$

   g $(\forall x)(\forall y)Q(x, y)$ **h** $(\forall y)(\forall x)Q(x, y)$ **i** $(\exists x)(\forall y)H(x, y)$

   j $(\exists x)(\exists y)H(x, y)$

4.2 ARGUMENTS AND VALIDITY

The most important part of mathematical logic as a system of logic is to provide the rules of inferences which play a central role in the general theory of the principle of reasoning. We are concerned here with a problem of decision, whether a certain chain of reasoning will be accepted as correct or incorrect on the basis of its form. By a chain of reasoning we mean a finite sequence of statements of which the last statement in the sequence, called the **conclusion** may be inferred from the initial set of statements called **premises.** The theory of inference may be applied to test the validity of an argument in everyday life.

ACTIVITY 4.6

1. What can be concluded about q, if p is true and $p \Rightarrow q$ is true?
2. If p and $p \wedge q$ have truth values T, what can be concluded about q?
3. If p and $p \vee q$ have truth values T, what can be said about q?

As you have seen from the above Activity, in order to come to the conclusion of the truth values of q, you evaluate the truth values of certain conditions called **hypotheses or premises**. Then you can find the truth value of another statement called the **conclusion.**

For example, in Activity 4.6 Question 2 given that p has truth value T and $p \wedge q$ has truth value T, you are asked to find the truth value of q. Actually, one can see from the rule for conjunction q must has truth value T; this is known as logical deduction, or argument form.

Definition 4.5

A **logical deduction (argument form)** is an assertion that a given set of statements $P_1, P_2, \ldots, P_n$, called hypotheses or premises yield another statement Q, called the **conclusion**. Such a logical deduction is denoted by:

$P_1, P_2, \ldots, P_n \vdash Q$ **Or**

$$\begin{array}{c} P_1 \\ P_2 \\ . \\ . \\ . \\ \underline{P_n} \\ Q \end{array}$$

Example 1 We can write the logical deduction in Activity 4.6 Question 2 as:

$p,\ p \wedge q \vdash q$ or $\begin{array}{c} p \\ \underline{p \wedge q} \\ q \end{array}$

An argument form is accepted to be either correct or incorrect (accepted or rejected) or valid or invalid (fallacy).

When do we say that an argument is valid or invalid?

Definition 4.6

An argument form $P_1, P_2, \ldots, P_n \vdash Q$ is said to be **valid** if Q is true, whenever all the premise $P_1, P_2, \ldots, P_n$, are true; otherwise it is **invalid.**

Example 2 Investigate the validity of the following argument forms.

a $p, p \Rightarrow q \vdash q$

Solution Now for the argument to be valid, we assume all the premises to be true and show that the conclusion is also true; otherwise it is invalid.

1 p is true ----------- premise

2 $p \Rightarrow q$ is true ------ premise

Therefore, q must be true from 1, 2 and rule for " $\Rightarrow$".

Therefore, the argument form $p, p \Rightarrow q \vdash q$ is valid.

You can use truth table to test validity as follows:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The premises p and $p \Rightarrow q$ are true simultaneously in row 1 only. Since in this case q is also true, the argument is valid.

b If you study hard, then you will pass the exam. You did not pass the exam.

Therefore, you did not study hard.

Solution:

Let p: you study hard.

q: you will pass the exam.

$\neg p$: you did not study hard.

$\neg q$: you did not pass the exam.

The argument form for b is, therefore written as,

$$\begin{array}{l} p \Rightarrow q \\ \underline{\neg q} \\ \neg p \end{array}$$

Thus to check the validity, you have the following reasoning:

1 $\neg q$ is true --------------premise

2 q is false -------------- using (1)

3 $p \Rightarrow q$ is true ---------- premise

4 p is false from (2) and (3), and rule of "$\Rightarrow$"

5 $\neg p$ is true from (4)

Therefore, the argument form is valid.

Alternatively, you can use the following truth table, to decide whether the argument is valid or not.

p	q	$\neg q$	$\neg p$	$p \Rightarrow q$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

The premises $p \Rightarrow q$ and $\neg q$ are true simultaneously in row 4 only. Since in this case $\neg p$ is also true, the argument is valid.

c $p \Rightarrow q, \neg q \Rightarrow r \vdash p$

Solution use the following truth table:

p	q	r	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow r$
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	T	F

The premises $p \Rightarrow q, \neg q \Rightarrow r$ are true in the 1^{st}, 2^{nd}, 5^{th}, 6^{th} and 7^{th} rows, but the conclusion p is false in the 5^{th}, 6^{th} and 7^{th} rows.

Therefore, the argument form is invalid.

Note that we can show whether an argument form is valid or invalid by two methods as illustrated by Example 2 above. One is by using a truth table and the other is without using a truth table. The proof provided without using a truth table, just by a sequence of reasoning, is called a **formal proof.**

Exercise 4.9

1 Decide whether each of the following argument forms is valid or invalid.

a $\neg p \Rightarrow q, q \vdash p$ **b** $p \Rightarrow \neg q,\ p,\ r \Rightarrow q \vdash \neg r$

c $p \Rightarrow q,\ \neg r \Rightarrow \neg q \vdash \neg r \Rightarrow \neg p$ **d** $p \Rightarrow q,\ q \vdash p$

e $p \vee q,\ p \vdash q$

2 For the following argument forms given in (I) and (II) below:

a Identify the premises and the conclusion.

b Use appropriate symbols to represent the statements in the argument.

c Write the argument forms using symbols.

d Check the validity.

I If the rain does not come, then the crops are ruined and the people will starve. The crops are not ruined or the people will not starve.

Therefore, the rain comes.

II If the team is late, then it cannot play the game. If the referee is here, then the team can play the game. The team is late.

Therefore, the referee is not here.

Rules of inferences

You have seen how to test the validity of arguments by using truth tables and formal proof. But in practice, testing the validity of an argument using a truth table becomes more difficult as the number of component statements increases. Therefore, in such cases, we are forced to use the formal proof. The formal proof regarding the validity of an argument relies on logical rules called **rules of inferences**. A formal proof consists of a sequence of finite statements comprising the premises and the consequence of the premises called **the conclusion.** The presence of each statement must be justified by a rule of inferences. It is obvious that we repeatedly apply these rules to justify the proof of complex arguments. Below are a few examples of some of these rules together with their classical names.

1 Modes Ponens

$$\begin{array}{c} P \\ P \Rightarrow Q \\ \hline Q \end{array}$$

2 Modes Tollens

$$\begin{array}{c} \neg Q \\ P \Rightarrow Q \\ \hline \neg P \end{array}$$

3 Principle of Syllogism

$$\begin{array}{c} P \Rightarrow Q \\ Q \Rightarrow R \\ \hline P \Rightarrow R \end{array}$$

4 Principle of adjunction

a
$$\begin{array}{c} P \\ Q \\ \hline P \wedge Q \end{array}$$

b
$$\begin{array}{c} P \\ \hline P \vee Q \end{array}$$

5 Principle of detachment

$$\begin{array}{c} P \wedge Q \\ \hline P,\ Q \end{array}$$

6	**Modes Tollendo ponens**	$\dfrac{\neg P \quad P \vee Q}{Q}$
7	**Principle of equivalence**	$\dfrac{P \Leftrightarrow Q \quad P}{Q}$
8	**Principle of conditioning**	$\dfrac{P}{Q \Rightarrow P}$

Let us see an example to illustrate how to use the rules of inferences in testing validity.

Example 3 Give a formal proof of the validity of the argument given below.

$P \wedge Q,\ (P \vee R) \Rightarrow S \vdash P \wedge S$

Proof:

1. $P \wedge Q$, has truth value T………….. premise.
2. $(P \vee R) \Rightarrow S$, has truth value T …… premise
3. P has truth value T ………..principle of detachment from (1).
4. $P \vee R$, has truth value T….. principle of adjunction (b) from (3)
5. S has truth value T………. Modes Ponens from (2) and (4).
6. $P \wedge S$ has truth value T….Principle of adjunction (a) from (3) and (5).

Therefore, the argument form $P \wedge Q,\ (P \vee R) \Rightarrow S \vdash P \wedge S$ is valid.

Exercise 4.10

1 Use the rules of inferences to test the validity of each of the following argument forms.

a $P \Rightarrow Q,\ R \Rightarrow P,\ R \vdash Q$ **b** $\neg P \wedge \neg Q,\ (Q \vee R) \Rightarrow P \vdash R$

c $P \Rightarrow \neg Q,\ P,\ R \Rightarrow Q \vdash \neg R$ **d** $\neg P \wedge \neg Q,\ (\neg Q \Rightarrow R) \Rightarrow P \vdash \neg R$

2 Given an argument form:

If a person stays up late tonight, then he/she will be dull tomorrow. If he/she does not stay up late tonight, then he/she will feel that life is not worth living. Therefore, either the person will be dull tomorrow or will feel that life is not worth living.

a Identify the premises and the conclusion.

b Use appropriate symbols to represent the statements in the argument.

c Write the argument form using symbols.

d Check the validity using rules of inferences.

Key Terms

arguments	**logical connectives (or logical operators)**
compound proposition	**open proposition (or open statement)**
contra positive of a conditional statement	**proposition (or statement)**
contradiction	**quantifiers; both existential and universal**
converse of a conditional statement	**rules of inferences**
equivalent compound propositions	**tautology**
invalid arguments	**valid arguments**

Summary

1. **Mathematical reasoning** is a tool to organize evidence in a systematic way through mathematical logic.
2. A sentence which has a truth value is said to be a **proposition** (or statement).
3. A sentence with one or more variables which becomes a statement on replacing the variable(s) by individual (s) is called an **open proposition** (or **open statement**).
4. The usual connectives in logic are: **or**, **and**, **not**, **if.... then** and **if and only if**.
5. A statement formed by joining two or more statements by a connective (or connectives) is called a **compound statement**.
6. A compound statement is a **tautology**, if and only if for every assignment of truth values to the component propositions occurring in it, the compound proposition always has truth value T. It is a **contradiction**, if the compound proposition always has truth value F.
7. We use the symbols $\exists$ and $\forall$ for the phrase "**there is**", (**existential quantifier**) and for the phrase "**for all**", (**universal quantifier**) respectively.
8. A logical deduction (argument form) is an assertion that a given set of statements $P_1, P_2, .., P_n$, called hypotheses or premises, yield another statement Q, called the **conclusion**.
9. To decide whether an argument is valid or invalid, we use a truth table or formal proof.
10. The formal proof regarding the validity of an argument relies on logical rules **called rules of inferences**.

Review Exercises on Unit 4

1 Which of the following compound propositions are tautologies, contradictions or neither.

a $(p \Rightarrow \neg q) \wedge (p \Rightarrow q)$ **b** $(\neg p \vee q) \Rightarrow (p \wedge \neg q)$

c $[(p \Rightarrow q) \vee (p \Rightarrow r)] \Leftrightarrow [p \Rightarrow (q \vee r)]$ **d** $(p \Rightarrow q) \Leftrightarrow \neg(\neg q \Rightarrow \neg p)$

2 Given $P(x)$: $\sqrt{x^2} = |x|$;

$Q(x)$: $x - 1 = 3$;

$R(x, y)$: $x + y = 0$

$T(x, y)$: $x + y = y$

Determine the truth value of each of the following, assuming that the universe is the set of real numbers.

a $(\exists x)\ P(x)$ **b** $(\forall x)\ P(x)$ **c** $(\exists x)\ Q(x)$

d $(\forall x)\ Q(x)$ **e** $(\exists x)(\forall y)\ R(x, y)$ **f** $(\forall x)\ (\exists y)\ R(x, y)$

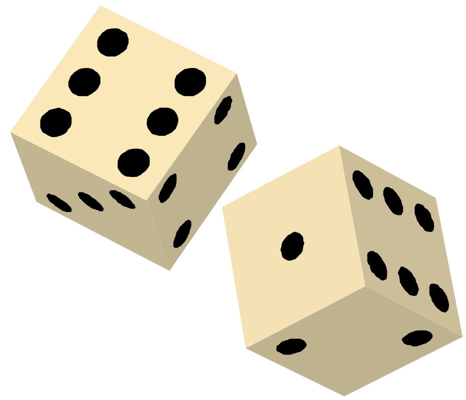
g $(\forall x)\ (\forall y)\ R(x, y)$ **h** $(\exists x)\ (\forall y)\ T(x, y)$ **i** $(\forall x)(\exists y) T(x, y)$

3 Check the validity of each of the following arguments.

a $\neg p \wedge q,\ (q \vee r) \Rightarrow p \vdash \neg r$ **b** $p \Rightarrow (q \vee r),\ \neg r,\ p \vdash q$

c If Mathematics is a good subject, then it is worth learning. Either the grading system is not fair or Mathematics is not worth learning. But the grading system is fair. Therefore, Mathematics is not a good subject.

Unit 5



STATISTICS AND PROBABILITY

Unit Outcomes:

After completing this unit, you should be able to:

- *know specific facts about types of data.*
- *know basic concepts about grouped data.*
- *know principles of counting.*
- *apply facts and principles in computation of probability.*

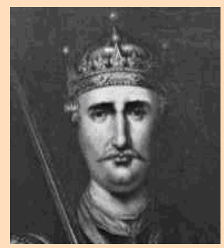
Main Contents

INTRODUCTION

The word statistics comes from the Italian word "statista" meaning stateman. It was used to signify the application of recorded data for purposes of the state. When statistics is used in its plural sense, it means a body of numerical facts and figures. The numerical facts are called statistical data, or simply data. When it is used in its singular form, **statistics** is a branch of mathematical science, and is concerned with the development and application of methods and techniques for the **collection**, **organization**, **analysis** and **interpretation** of quantitative data. We will confine ourselves to this second meaning of statistics through this unit.

HISTORICAL NOTE

William I of England (1027-1087)



In December, 1085, William the Conqueror decided to commission an inquiry into the ownership, extent and values of the land of England to maximize taxation. This unique survey is known to history as "The Domesday Book" and is considered to be the first statistical abstract of England.

OPENING PROBLEM

The following data are the results of 20 students in a Mathematics final exam (out of 100):

75	52	80	71	60	45	90	58	63	49
83	69	74	50	92	78	59	68	70	82

a Arrange the data in increasing order.

b Group the data into five classes.

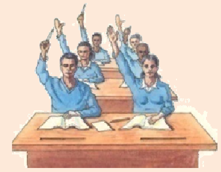
c Draw a histogram of the grouped data.

5.1 STATISTICS

Recall that you have studied the basics of statistics in Grade 9, including its meaning, importance and purpose. You also have discussed presentation of data using different forms such as a histogram, measures of central tendency, and measures of dispersion of data. The work in this grade will begin with discussing types of data.

5.1.1 Types of Data

ACTIVITY 5.1



1 Classify the following data as qualitative or quantitative:

a beauty of a picture **b** size of your shoe

c type of a car **d** number of children living in a house

e colour of your skin **f** blood type(group)

2 Classify the following variables as discrete or continuous:

a size of a shirt **b** number of members of a football club

c price of a kilo of sugar **d** number of rooms in a house

e heights of students in a class **f** life-time of an electric bulb

The first step in applying statistical methods is the collection of data; this is the process of obtaining counts or measurements. The data obtained can be classified into two types: qualitative or quantitative data.

Definition 5.1

Qualitative data is obtained when a given population or sample is classified in accordance with an attribute that cannot be measured or expressed in numbers, while **Quantitative data** is that obtained by assigning a real number to each member of the population, under study.

Example 1 Classify the following data as qualitative or quantitative:

Honesty, height, weight, intelligence, income, efficiency, width, sex, pressure, distance, religion, social status.

Solution: Honesty, intelligence, efficiency, sex, religion and social status are qualitative, while height, weight, income, width, pressure and distance are quantitative.

[If I.Qs (intelligent quotients) are used to measure intelligence, then it will be quantitative.]

Definition 5.2

A number, which is used to describe the attribute and which can take different values is called a **variable**.

For example, in your class the height, weight or age of different individuals varies, and can be expressed in numbers. Therefore, these quantities (height, weight, age,…) are variables.

✍**Note:**

Variables are denoted by letters such as x, y, z,...

A variable may be either discrete or continuous.

Definition 5.3

A **Discrete Variable** is one which takes only whole number values. It is usually obtained by counting. There is a gap between consecutive values i.e. it varies only by finite jumps. A **Continuous Variable** is one which takes all real values between two given real values.

Example 2 Which of the following are discrete variables? Which are continuous?

Number of students in a class, weight of students, length of a road, number of chairs in a room, temperature of a room and number of houses along a street.

Solution: Number of students in a class, number of chairs in a room and number of houses along a street are discrete. They can have whole number values only.

On the other hand, weight of students, length of a road and temperature of a room are continuous variables. They can take fractional or decimal values. For instance, weight of students could be given by values like 50.1kg, 49.73kg; length of a road could be given by values like 6.5km, 2.63km, while temperature of a room could be given by values like 20^{o}C, 14.5^{o}C.

Group Work 5.1

Do the following in groups.

1 Suppose data is collected about a set of people, as given below:

a	gender	**b**	religion	**c**	educational qualification
d	number of children	**e**	income	**f**	shoe size
g	height	**h**	weight	**i**	nationality

Classify each of them as qualitative, discrete quantitative or continuous quantitative data.

2 Consider the following example: “Weight” for most humans is measured on the following scale (in kilograms).

0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150

Following the example, design suitable scales for the following.

a height (humans) **b** top speed (cars) **c** monthly income

5.1.2 Introduction to Grouped Data

Definition 5.4

A **Frequency distribution** is a table which shows the list of all values of data obtained and the number of times these values occur (frequency). The raw data obtained will be organized and summarized into a **grouped frequency distribution table** for the purpose of summarizing a large amount of data.

Example 3 Consider the following data. It represents the number of patients that a doctor visits per day for 150 working days.

3	2	6	2	6	5	22	3	1	10	2	6	6	11	8
5	9	7	2	5	1	5	4	9	7	11	3	14	1	4
25	19	8	2	5	8	10	16	15	5	6	8	4	12	13
7	8	3	6	6	21	6	9	4	5	6	8	29	9	23
6	6	22	8	11	23	8	5	9	6	5	18	7	4	5
8	7	5	10	16	11	13	1	7	3	18	5	8	11	5
2	18	0	16	4	9	8	5	9	17	3	11	20	6	28
7	9	5	19	12	1	10	3	0	7	8	17	5	9	7
13	18	8	7	8	7	7	13	9	5	20	10	6	22	1
14	7	20	1	9	4	6	24	17	6	4	6	14	4	4

Solution The data given is raw data or ungrouped data. To summarize the raw data into a grouped frequency distribution, follow these steps:

Steps to prepare a grouped frequency distribution table

1 Determine number of classes required (usually between 5 and 20).

2 Approximate the interval of each class or class width using the following formula

$$\text{Class interval} = \frac{\text{maximum value} - \text{minimum value}}{\text{number of classes required}}$$

To prepare the frequency distribution, first you decide the number of classes. For this case, let the number of classes be 5.

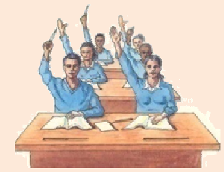
$$\text{Class interval } (w) = \frac{29 - 0}{5} = 5.8 \quad (\textit{From the formula for class interval})$$

Note:

From the formula, the class interval, w, is calculated as 5.8. For practical purposes, it will be convenient to choose the class interval to be a whole number. For this case, you can take class interval as 6. (This is obtained by rounding 5.8 to the nearest whole number). Therefore $w = 6$ (See the grouped frequency distribution below).

Number of patients (class limit)	Tally	Number of visiting days (f)
0 – 5	卌 卌 卌 卌 卌 卌 卌 卌 卌 ||||	49
6 – 11	卌 卌 卌 卌 卌 卌 卌 卌 卌 卌 卌 卌 卌 |	66
12 – 17	卌 卌 卌 |	16
18 – 23	卌 卌 卌	15
24 – 29	||||	4
	Total	150

ACTIVITY 5.2



1 What is the frequency of the 2nd class?

2 What is the frequency of the 5th class?

In the above frequency distribution, you are considering frequencies of each class. But, in reality you may be interested to know about other issues such as how many days the doctor visited fewer than 8 patients. To answer such a question, the frequency distribution given above may not always be suitable. For such a purpose, you need to construct what is called a cumulative frequency distribution.

A cumulative frequency distribution is constructed by either successively adding the frequencies of each class called "less than cumulative frequency" or by subtracting the frequency of each class from the total successively called "more than cumulative frequency".

The cumulative frequency distribution of the above data of patients that a doctor visits per day is as follows.

Number of patients (class limit)	Tally	Number of visiting days (f)	Cumulative frequency
0 – 5	卌 卌 卌 卌 卌 卌 卌 卌 卌 ||||	49	49
6 – 11	卌 卌 卌 卌 卌 卌 卌 卌 卌 卌 卌 卌 卌 |	66	115
12 – 17	卌 卌 卌 |	16	131
18 – 23	卌 卌 卌	15	146
24 – 29	||||	4	150
	Total	**150**	

Note that the above frequency distribution is for a discrete variable.

Definition 5.5

The first and the last elements of a given class interval are called **class limits**.

Example 4 For the above table, give the lower and upper class limits for the second and the fourth classes.

Solution: For the second class 6 – 11, 6 is called the lower class limit and 11 is called the upper class limit, while the lower limit and the upper limit of the fourth class are 18 and 23 respectively.

Exercise 5.1

1 Describe whether each of the following are qualitative or quantitative.

- **a** Beauty of a student
- **b** Volume of water in a barrel
- **c** Score of a team in a soccer match
- **d** Neatness of our surrounding

2 Identify whether each of the following is discrete or continuous.

- **a** Yield of wheat in quintals
- **b** Rank of students by examination results
- **c** Volume of water in a barrel
- **d** Sex of a student

3 The following are scores of 40 students in a statistics exam.

50	72	56	31	48	33	56	54	41	35
22	76	32	66	56	38	48	36	44	46
36	49	51	59	62	41	36	50	41	42
50	50	49	60	36	46	42	42	47	62

Prepare a grouped frequency distribution, using 7 classes. Answer the following questions.

- **a** What is the class interval?
- **b** What is the lower class limit of the second class?
- **c** What is the upper class limit of the second class?
- **d** What is the frequency of the first class?

4 The following are weights (in kg) of 50 patients in a hospital.

70	62	58	42	18	33	24	54	64	29
12	76	28	54	59	42	53	24	48	36
42	59	64	46	62	52	24	42	48	58
60	54	39	56	36	78	16	26	58	62
34	18	22	28	62	38	46	53	62	37

Prepare a grouped frequency distribution, using 6 as class width. Answer the following questions.

a How many classes do we have?

b Determine the cumulative frequency distribution?

c How many patients do have their weight less than 48kgs?

d What is the frequency of the fourth class?

e What is the cumulative frequency at the seventh class?

Definition 5.6

1 The average of the upper and lower class limit is called the **class mark** or **class midpoint**.

$$\text{Class mark} = \frac{\text{lower class limit} + \text{upper class limit}}{2}$$

2 The **correction factor** is half the difference between the upper class limit of a class and the lower class limit of the subsequent class.

✍Note:

The class mark serves as representative of each data value in a class (or the class itself).

Example 5 For the following distribution which shows the test scores of 60 students in a mathematics test corrected out of 100, give the correction factor.

Score (Class limit)	Number of students (Frequency) (*f*)
1 – 25	5
26 – 50	10
51 – 75	30
76 – 100	15

Solution: In this distribution, the correction factor is

$$\frac{1}{2}(26-25)=0.5 \text{ or } \frac{1}{2}(51-50)=0.5$$

Why do you need the correction factor?

Previously, you saw that a cumulative frequency distribution of discrete values may help answer some questions. But, there could be more questions to answer. For example, in Example 5 above, suppose you are asked '*To which class does a mark of 9.5 belong?* Or, *how many students have scored less than 9.5*?' To solve such problems, you have to smoothen the distribution and fill the gaps. In order to smoothen, you add the correction factor to the upper limits of each class and subtract from the lower class limits of each class to get what are called **class boundaries**.

Then the class 25.5–50.5 includes variable values that are 25.5 and above, but below 50.5.

Group Work 5.2

Do the following in groups.

1 Consider the frequency distribution table considered in Example 5 above.

Copy the table and insert columns which show class boundaries, class mid points and cumulative frequency and fill them in.

2 100 students have taken a mathematics test and the teacher has organized the data into the following table:

Test mark	1– 5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	41–45	46–50
Frequency	1	2	11	17	25	18	13	6	3	4

Using what you have learned in grade 9, draw a histogram of the data.

Steps to construct a frequency distribution:

1 Find the highest and lowest values.

2 Find the range (i.e., highest value – lowest value).

3 Select the number of classes desired.

4 Find the class interval by dividing the range by the number of classes and rounding up.

5 Select a starting point (usually the lowest value); add the class interval to get the lower limits.

6 Find the upper class limits.

7 Tally the data.

8 Find the frequencies.

9 Find the cumulative frequency.

Exercise 5.2

1 A teacher in a school has given a project to her students to make a survey of the size of two kinds of trees in a forest nearby. The following is the frequency table that the students made about the circumference of 100 randomly selected trees of each of two kinds A and B.

Circumference (cm)	Tree type A (*f*)	Tree type B(*f*)
1–20	5	4
21–40	15	4
41–60	25	12
61–80	19	8
81–100	22	22
101–120	7	26
121–140	5	18
141–160	2	6

a What is the class interval?

b What is the lower class limit of the second class?

c What is the upper class limit of the second class?

d What is the frequency of the first class?

e Complete the following table about Tree type A.

Circumference (cm)	Class Boundaries	Class midpoint	Tree type A (*f*)
1–20			5
21–40			15
41–60			25
61–80			19
81–100			22
101–120			7
121–140			5
141–160			2

f Make a similar table for Tree type B.

g Draw histograms to illustrate both frequency distributions.

2 The following are yield in quintals of wheat harvested by thirty farmers per hectare.

42	39	26	18	22	52	24	12	24	32
48	33	29	56	36	24	16	32	21	78
16	28	30	16	62	38	14	19	30	54

Prepare a grouped frequency distribution, using 11 classes. Answer the following questions.

a What is the lower class limit for the third class?

b What is the lower class boundary for the seventh class?

c Determine the correction factor for this frequency distribution.

d What is the class mark of the second class?

e Find the difference between the class marks of the eighth and ninth classes.

5.1.3 Measures of Location for Grouped Data

When you want to make comparisons between groups of numbers, it is good to have a single value that is considered to be a good representative of each group. One such value is the average of the group. Averages are also called **measures of location or measures of central tendency**. The most commonly used measures of central tendency are **Mean** (or **Arithmetic mean**), **Median**, **Mode**, **Quartiles**, **Deciles** and **Percentiles**.

In Grade 9, you learned how to find the mean, median and mode for ungrouped data. In this section, we will focus onto grouped frequency distributions.

First, let us recall the summation notation. Let $x_1, x_2 \ldots, x_n$ be a number of values where n is the total number of observations with x_i the ith observation.

The symbol $\sum_{i=1}^{n} x_i$ is called sigma or the **summation notation** and i is called an **index**, with I = 1 the starting index and I = n the ending index.

Thus $\sum_{i=1}^{n} x_i = x_1 + x_2 + \ldots + x_n$.

The mean

Definition 5.7

The mean $\overline{x}$ of a set of data is equal to the sum of the data items divided by the number of items contained in the data set.

If $x_1, x_2, x_3, \ldots x_n$ are n values, their mean is given by

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

If $x_1, x_2, \ldots x_n$ is a set of data items, with frequencies $f_1, f_2, \ldots, f_n$, respectively, then their mean is given by

$$\overline{x} = \frac{f_1x_1 + f_2x_2 + \ldots + f_nx_n}{f_1 + f_2 + \ldots + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{\sum_{i=1}^{n} f}$$

Example 6 Calculate the mean of 7, 6, 2, 3, 8.

Solution: $\overline{x} = \frac{7+6+2+3+8}{5} = \frac{26}{5} = 5.2$

Example 7 Consider the following values which show the number of radios sold by an electronics shop for 25 days.

7, 7, 2, 6, 7, 10, 8, 10, 2, 7, 10, 7, 2, 7, 6, 10, 6, 7, 8, 7, 6, 7, 10, 6, 10

a Prepare a frequency distribution table.

b Find the mean number of radios sold from the frequency distribution table.

Solution

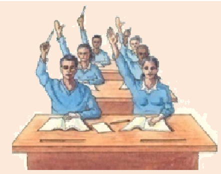
a From the above raw data , you may have found the following frequency distribution table which shows the number of radios sold by the shop for 25 days.

x	2	6	7	8	10
f	3	5	9	2	6

b We use the above formula to find the mean from the frequency distribution table.

$$\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{3\times2+5\times6+9\times7+2\times8+6\times10}{3+5+9+2+6} = \frac{6+30+63+16+60}{25} = \frac{175}{25} = 7$$

ACTIVITY 5.3



1. A group of 5 water tanks in a farm have a mean average height of 4.7 metres. If a sixth water tank with a height of 2.91 metres is erected, what is the new mean average height of the water tanks?
2. One group of 8 students has a mean average score of 67 in a test. A second group of 17 students has a mean average score of 81 in the same test. What is the mean average of all 25 students?
3. Write a general formula to find the combined mean of two groups of data and explain.

Mean for grouped data

The procedure for finding the mean for grouped data is similar to that for ungrouped data, except that the mid points of the classes are used for the x values.

Example 8 Calculate the mean average of this grouped frequency table for students' test scores.

Mark	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40	41-45	46-50
f	1	2	17	25	11	13	18	5	4	4

Solution: If you have to use what you know so far to calculate the mean, we need to know the total number of students that took the test and the total number of marks that they scored.

The total number of students is 100, but we have a problem when it comes to the total number of marks. Since you have grouped data, you cannot obtain individual marks. For instance, 13 students scored between 26 and 30. But, there is no way you can tell the total mark of the 13 students.

The way out of this problem is to approximate each student's mark by the middle mark of the class interval, as in the following table:

Mark	1-5	6-10	11-15	16-20	21-25	26-30	31-35	36-40	41-45	46-50
Mid Value (x_c)	3	8	13	18	23	28	33	38	43	48
f	1	2	17	25	11	13	18	5	4	4
$f \times x_c$	3	16	221	450	253	364	594	190	172	192

Now, total number of students = 100; Total marks (approximate) = 2455

Therefore, approximate mean = $\frac{2455}{100} = 24.55$.

✍**Note:**

Remember that this mean is an approximation based on the assumption that each class is represented by a midpoint without much loss of accuracy. In calculating the mean of a grouped distribution, each class is represented by its class mark (class midpoint).

Steps to find the mean from a grouped distribution

From a grouped frequency distribution

1 Find the class mark (mid point) x_c of each class, by $\frac{\text{lower class limit + upper class limit}}{2}$.

2 Multiply x_c by its corresponding frequency and add.

3 Divide the sum obtained in step 2 by the sum of the frequencies.

$$\bar{x} = \frac{f_1 x_{c_1} + \ldots + f_n x_{c_n}}{f_1 + f_2 + \ldots + f_n} = \frac{\sum f_i x_{c_i}}{\sum f_i}$$

Example 9 The following is the age distribution of 20 students in a class. Find the mean age of these students.

Age (in years)	Class mid point (x_c)	Number of students (f)	fx_c
14 – 18	16	2	32
19 – 23	21	7	147
24 – 28	26	6	156
29 – 33	31	5	155

$\sum f = 20$ $\sum fx_c = 490$

Solution: $\bar{x} = \frac{\sum fx_c}{\sum f} = \frac{490}{20} = 24.5$ years

The procedure for finding the mean for grouped data assumes that all of the raw data values in each class are equal to the class mark of the class. In reality, this is not true. However, using this procedure will give us an acceptable approximation of the mean, since some values usually fall above the class mark and others fall below the class mark for each class.

Exercise 5.3

1 The following frequency distribution tables represent score and age of students. Find the mean for each of them.

a

Marks	Frequency
10 – 12	4
13 – 15	7
16 – 18	10
19 – 21	13
23 – 25	16

b

Age	Frequency
13 – 15	6
16 – 18	6
19 – 21	3
22 – 23	2

2 Forty-six randomly selected light bulbs were tested to determine their life time (in hours) and the following frequency distribution was obtained. Find the mean hour of life time.

Life time (hrs)	Frequency
54 – 58	2
59 – 63	5
64 – 68	10
69 – 73	14
74 – 78	10
79 – 83	5

3 The following are quintals of fertilizer distributed to fifty farmers.

24	19	26	28	29	25	32	22	24	18
32	13	31	26	18	18	26	14	24	24
28	32	23	16	24	19	34	31	13	36
16	23	32	41	34	24	31	23	18	42
6	8	24	26	34	18	32	19	28	14

a Find the average number of quintals of fertilizer distributed to the farmers from the raw data.

b Prepare discrete frequency distribution and calculate the mean.

4 Using the data given in Question 3 prepare two grouped frequency distributions, using 6 and 9 classes. Answer the following questions.

i Find the mean of each.
ii Are the four means you calculated equal?
iii Write your generalizations.

The median (*md*)

You should remember that median of a set of data is the middle number when the data is arranged in either increasing or decreasing order of magnitude. It is a half way point in a data set, when the data is arranged in order (called a data array). The median will be a value in the data or will fall between two values.

Example 10

a The following data shows the age to the nearest year of 7 students in a class. What will be the median of this age distribution?

6, 8, 5, 6, 10, 7, 3.

b Find the median from the following data.

60, 63, 59, 72, 50, 49.

Solution

a Arranging in an increasing order gives 3, 5, 6, 6, 7, 8, 10.

Since the number of observations is 7 and this number is odd, therefore,

$$md = \left(\frac{n+1}{2}\right)^{th} \text{item} = \left(\frac{7+1}{2}\right)^{th} \text{item} = 4^{th} \text{ item which shows the median is 6.}$$

b First you have to arrange in increasing order giving,

49, 50, 59, 60, 63, 72.

Since $n = 6$, which is even, you will use the second formula

$$md = \frac{\left(\frac{n}{2}\right)^{th} \text{item} + \left(\frac{n}{2}+1\right)^{th} \text{item}}{2} = \frac{\left(\frac{6}{2}\right)^{th} \text{item} + \left(\frac{6}{2}+1\right) \text{item}}{2}$$

$$md = \frac{3^{rd} \text{ item} + 4^{th} \text{ item}}{2} = \frac{59+60}{2} = \frac{119}{2} = 59.5$$

Exercise 5.4

1 Consider the following data which shows the amount of milk in litres sold by a farmer in one month.

5, 6, 7, 6, 8, 10, 10, 8, 7, 6, 5, 4, 8, 7, 6, 5, 4, 8, 8, 7, 6, 5, 6, 7, 8, 10, 8, 7,6,5

a Find the median from the raw data.

b Prepare a frequency distribution table.

Hint:- Your table may help you to arrange the values in an increasing order.

2 Find the median of the following distribution.

x	2	5	7	8	10
f	3	4	9	3	6

3 The Bills paid (in Birr) for electric consumption by Ato Abebe in the last 12 months is as follows.

52, 68, 57, 96, 78, 48, 103, 82, 71, 62, 51, 24

a Find the median of Bills paid for the electric consumption.

b Calculate the mean and compare it with the median.

4 The following data shows score of fifty students in Mathematics exam

14	19	16	13	14	19	13	18	14	15
17	18	14	17	18	18	14	14	16	17
15	14	15	16	15	17	14	15	18	14
16	17	16	14	14	14	15	17	14	17
14	16	14	15	15	16	16	14	15	16

a Find the median from the raw data.

b Prepare a discrete frequency distribution table and calculate the median.

Median for grouped data

So far, you have seen how to find the median from raw data and, from the above Exercise, you should have been able to find the median from a single valued frequency distribution table. In the next part, you will see the steps to find the median of a grouped frequency distribution.

Steps to find the median of a grouped frequency distribution

1. Prepare a cumulative frequency distribution.
2. Find the class where the median is located. It is the lowest class for which the cumulative frequency equals or exceeds $\frac{n}{2}$.
3. Determine the median by the formula $md = B_L + \left(\frac{\frac{n}{2} - cf_b}{f_c} \right) i$

where,

B_L = Lower boundary of the class containing the median (median class)

n = total number of observations $\left(\sum f \right)$.

cf_b = the cumulative frequency in the class preceding ("coming before") the class containing the median.

f_c = the number of observations (frequency) in the class containing the median.

i = the size of the class interval.(i.e. width of the median class)

Example 11 The following is the height of 30 students in a class. Find the median height.

Height (in cm)	Number of students (f)
140 – 145	7
146 – 151	9
152 – 157	8
158 – 163	4
164 – 169	2

✍ **Note:**

First use the correcting factor to prepare a cumulative frequency table.

The correcting factor is $\frac{146-145}{2} = 0.5$. (*uniform for all classes*)

From this, you can prepare the class boundary column and the cumulative frequency column as follows.

height (in cm)	height (in cm) (class boundaries)	f	cf (Cumulative frequency)
140 – 145	139.5 – 145.5	7	7
146 – 151	145.5 – 151.5	9	16 = 7 + 9
152 – 157	151.5 – 157.5	8	24 = 16 + 8
158 – 163	157.5 – 163.5	4	28 = 24 + 4
164 – 169	163.5 – 169.5	2	30 = 28 + 2
	Total	30	

The median class is that class containing the $\left(\frac{30}{2}\right)^{th}$ item = 15^{th} item. It is in the 2^{nd} class.

Therefore, the median class is 145.5 – 151.5.

Thus, $B_L = 145.5$, $\frac{n}{2} = 15$, $f_c = 9$, $i = 151.5 - 145.5 = 6$, $cf_b = 7$

Therefore, $md = B_L + \left(\frac{\frac{n}{2} - cf_b}{f_c}\right)i = 145.5 + \left(\frac{15-7}{9}\right)6$

$= 145.5 + 5.333$

$= 150.83$

The median height is 150.83 cm.

Exercise 5.5

1 The following data shows Age of forty students in a class.

17	19	14	17	18	16	19	13	19	17
13	14	16	13	14	17	14	16	18	15
16	13	15	12	14	13	14	17	18	15
18	16	17	20	16	17	19	21	17	16

a Find the median from the raw data.

b Construct a grouped frequency distribution, with 5 classes.

c Find the median from the frequency distribution table.

2 Calculate the median of each of the following sets of data about students in a class.

a

Daily income (in Birr)	Number of students
10 – 14	4
15 – 19	11
20 – 24	17
25 – 29	16
30 – 34	8
35 – 39	4

b

Marks	Number of students
20 – 29	2
30 – 39	12
40 – 49	15
50 – 59	10
60 – 69	4
70 – 79	4
80 – 89	3

3 The amounts of drops of water in drip irrigation were registered from 80 sample drip holes in one day and the data are as follows.

77	99	104	87	108	86	91	87	92	77	103	104	96	92
92	97	79	97	101	95	113	85	84	112	78	73	86	77
107	67	88	76	77	87	114	97	102	101	98	105	67	67
94	118	79	68	64	103	87	97	73	92	78	95	86	99
87	76	99	112	68	103	98	63	101	101	76	67	79	84
87	116	102	81	76	88	98	93	82	78				

a Find the median from the raw data.

b Construct a grouped frequency distribution, with 10 classes and find the median.

4 Calculate the median of the following sets of data about score of students in an exam.

Score of students	Number of students
1 – 7	2
8 – 14	5
15 – 21	7
22 – 28	12
29 – 35	7
36 – 42	5
43 – 49	2
Total	40

a Find the mean and median score of the students.

b Compare the mean and the median.

The mode (m_o)

In statistics, the word mode represents the most frequently occurring value in a data set.

Definition 5.8

The **Mode** of a set of data is the value in the data which appears most frequently in the set of values.

Example 12 Find the mode of each of the following.

a 2, 5, 6, 5, 4, 2, 3, 2.

b 2, 3, 4, 8, 9

c 4, 8, 7, 4, 8, 2, 3

d

x	10	16	17	20	22	26
f	4	2	4	3	4	3

Solution:

a In this observation, the most frequent value is 2. Therefore, the mode is $m_o = 2$ since it appears three times. This data has only one mode and is called **unimodal**.

b Every member appeared only once. Hence there is no mode for this distribution.

c Here both 4 and 8 appear twice but the rest appear only once. Hence the modes are 4 and 8. This distribution has two modes. Such distributions are said to be **bimodal**.

d Three values 10, 17 and 22 all appear 4 times. Hence the modes are 10, 17 and 22. Distributions that have more than two modes are called **multimodal**.

Exercise 5.6

1 Determine the mode of each of the following data sets.

a

x	2	5	7	8	10
f	3	4	9	2	6

b

x	7	10	12	15
f	6	4	6	3

c 8, 12, 7, 9, 6, 18

d 7, 7, 10, 12, 10, 12

2 The following represent days in a month at which salary was paid for forty-two consecutive months.

22	27	26	24	23	25	28	27	26	23	25	24	27	26
25	27	28	25	26	27	27	24	27	26	25	27	26	27
23	22	27	28	27	29	27	23	27	24	26	27	27	26

a What is the mode of this data?

b At which date is salary paid mostly?

3 In electing student representative, there were three candidates: Abebe, Helen and Mahder. The following result was summarized.

Candidate	Abebe	Helen	Mahder
Number of votes	7	5	8

a What is the mode vote?

b Who must be elected? Why?

4 The following data represents shoe sizes of different models of shoes displayed in a boutique.

39	40	40	41	39	40	39	41
39	39	42	39	43	39	42	

a Determine the mode shoe size in the shop?

b What does this mode describe?

Mode of grouped data

✍Note:

Before we find any mode(s) that might exist, check the following points:

1 The class interval of all classes should be equal (Uniform class interval).

2 We need a column of class boundaries which can be obtained from the class limits.

Steps to calculate the modal value from grouped data

1 Identify the modal class. It is the class with the highest frequency.

2 Determine the mode using the following formula: Mode = $M_o = B_L + \left(\frac{d_1}{d_1 + d_2}\right)i$

Where B_L = lower class boundary of the modal class.

d_1= the difference between the frequency of the modal class and the frequency of the preceding class (pre-modal class).

d_2= the difference between the frequency of the modal class and the frequency of the subsequent class (next class).

i = size of the class interval.

Example 13 The following table gives the age distribution in a certain class. Compute the modal age (in years).

Age	f
10 – 14	7
15 – 19	6
20 – 24	10
25 – 29	2

Solution The modal class is the 3rd class because its frequency is the largest.

$B_L = 19.5$, $d_1 = 10 - 6 = 4$, $d_2 = 10 - 2 = 8$, $i = 24 - 19 = 5$

$$m_o = 19.5 + \left(\frac{4}{4+8}\right)5 = 19.5 + \frac{20}{12} = 19.5 + 1.67 = 21.17 \text{ years.}$$

Exercise 5.7

1 Find the mode for each of the following distributions.

a 5, 7, 8, 20, 15, 8, 7, 8, 20, 8.

b 8, 9, 12, 5.

c 10, 2, 5, 8, 12, 9, 9, 5, 9, 8, 7, 6, 1, 3, 8.

d

v	4	6	8	10	11
f	5	3	7	7	4

e

Marks	0–9	10–19	20–29	30–39	40–49
Frequency	12	18	27	20	17

2 The daily profits (in Birr) of 100 shops are distributed in the following table. Find the modal value.

Profit	1–100	101–200	201–300	301–400	401–500	501–600
No of shops	12	18	27	20	17	6

3 The following is a distribution of the size of farms (in 1000 m^2) in a woreda. Find the mode of the distribution.

Size of farm	5–14	15–24	25–34	35–44	45–54	55–64	65–74
No of farms	8	12	17	29	31	5	3

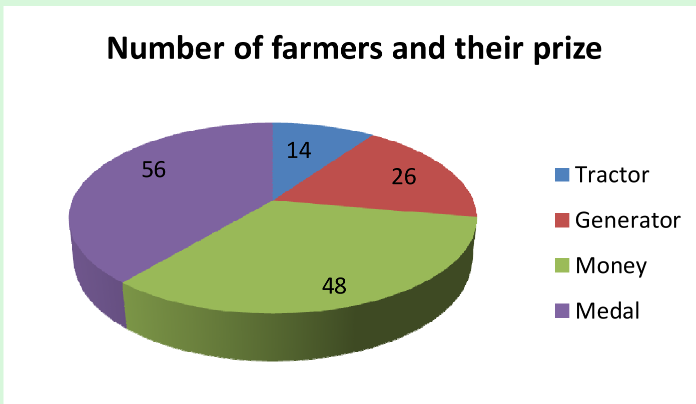
4 The amounts of drops of water in drip irrigation were registered from 80 sample drip holes in one day and the data are as follows.

77	99	104	87	108	86	91	87	92	77
103	104	96	92	92	97	79	97	101	95
113	85	84	112	78	73	86	77	107	67
88	76	77	87	114	97	102	101	98	105
67	67	94	118	79	68	64	103	87	97
73	92	78	95	86	99	87	76	99	112
68	103	98	63	101	101	76	67	79	84
87	116	102	81	76	88	98	93	82	78

a Find the mode from the raw data.

b Construct a grouped frequency distribution, with 10 classes and find the mode.

5 The number of farmers who got a prize for their productivity and the type of prize they got is given as follows.



Determine the mode prize.

Quartiles, deciles and percentiles

The median divides a distribution into two equal halves. There are other measures that divide the data into four, ten and a hundred equal parts. These values are called **quartiles**, **deciles** and **percentiles**, respectively.

These measures, which are recognized as measures of location, will be discussed for both ungrouped and grouped data.

Quartiles, deciles and percentiles for ungrouped data

1 Quartiles

Quartiles are values that divide a set of data into four equal parts. There are three quartiles, namely, Q_1, Q_2 and Q_3.

To calculate quartiles, follow these steps.

Steps to calculate quartiles for ungrouped data

1 Arrange the data in increasing order of magnitude.

2 If the number of observations is:

a odd, $Q_k = \left(\frac{k(n+1)}{4}\right)^{\text{th}}$ item.

b even, $Q_k = \left(\frac{\left(\frac{kn}{4}\right)+\left(\frac{kn}{4}+1\right)}{2}\right)^{th}$ item.

Example 14 Find Q_1 and Q_3 for the following data.

25, 38, 42, 46, 31, 29, 21, 9, 5.

Solution Arranging in increasing order of magnitude, we get,

5, 9, 21, 25, 29, 31, 38, 42, 46.

$Q_1 = \frac{1(9+1)}{4} = (2.5)^{\text{th}}$ item. What does this mean?

Q_1 lies half way between the 2^{nd} and 3^{rd} items.

Therefore, $Q_1 = 2^{nd}$ item + $\frac{1}{2}(3^{rd}\text{ item } - 2^{nd}\text{ item}) = x_2 + \frac{1}{2}(x_3 - x_2)$

$= 9 + \frac{1}{2}(21-9) = 9+6 = 15$ or $Q_1 = \frac{9+21}{2} = 15$

$Q_3 = \left(\frac{3(n+1)}{4}\right)^{th}$ item $= \left(\frac{3 \times 10}{4}\right)^{th}$ item $= (7.5)^{th}$ item.

It is half of the way between the 7^{th} (x_7) and 8^{th} (x_8) items.

Therefore, $Q_3 = x_7 + 0.5\,(x_8 - x_7) = 38 + 0.5\,(42 - 38)$

$= 38 + 2 = 40$

or $Q_3 = \frac{38+42}{2} = 40$

2 Deciles

Deciles are values that divide a set of data into ten equal parts. There are nine deciles, namely, D_1, D_2 , D_3, D_4, D_5, D_6, D_7, D_8, D_9.

To calculate deciles, follow these steps.

Steps to calculate deciles for ungrouped data

1 Arrange the data in increasing order of magnitude.

2 If the number of observations is:

a odd, $D_i = \left(\frac{i(n+1)}{10}\right)^{\text{th}}$ item.

b even, $D_i = \left(\frac{\left(\frac{in}{10}\right)+\left(\frac{in}{10}+1\right)}{2}\right)^{th}$ item.

Example 15 Find D_2 and D_7 for the following data: 25, 38, 42, 46, 50, 31, 29, 21, 9, 5.

Solution Arranging in increasing order of magnitude, we get,

5, 9, 21, 25, 29, 31, 38, 42, 46, 50.

$$D_2 = \left(\frac{\left(\frac{2(10)}{10}\right)+\left(\frac{2(10)}{10}+1\right)}{2}\right)^{th} \text{item} = \left(\frac{2+3}{2}\right)^{th} \text{item} = 2.5^{th}\text{item} = 15$$

$$D_7 = \left(\frac{\left(\frac{7(10)}{10}\right)+\left(\frac{7(10)}{10}+1\right)}{2}\right)^{th} \text{item} = \left(\frac{7+8}{2}\right)^{th} \text{item} = 7.5^{th}\text{item} = 40$$

3 Percentiles

Percentiles are values that divide a data set into a hundred equal parts. There are ninety nine percentiles, namely, P_1, P_2, … P_{99}.

Percentiles are not the same as percentages. If a student gets 85 correct answers out of a possible 100, he obtains a percentage score of 85. Here there is no indication of his position with respect to other students.

On the other hand if a score of 85 corresponds with the 96^{th} percentile, then this score is better than 96% of the students under consideration. Were your average and percentile in your grade eight exams the same?

To calculate percentiles, do the following:

Steps to calculate percentiles for ungrouped data

1 Arrange the data in increasing order of magnitude.

2 If the number of observations is:

a odd, $P_t = \left(\frac{t(n+1)}{10}\right)^{\text{th}}$ item.

b even, $P_t = \left(\frac{\left(\frac{tn}{100}\right) + \left(\frac{tn}{100}+1\right)}{2}\right)^{th}$ item.

Example 16 Find P_{42} and P_{75} for the following data.

25, 38, 42, 46, 50, 31, 29, 21, 9, 5.

Solution Arranging in increasing order of magnitude, we get,

5, 9, 21, 25, 29, 31, 38, 42, 46.

$$P_{42} = \left(\frac{42(n+1)}{100}\right)^{th} \text{ item } = \left(\frac{42 \times 10}{100}\right)^{th} \text{ item } = 4.2^{th} \text{ item}$$

Hence P_{42} is between the 4th and 5th item, i.e. $x_4 + 0.2\ (x_5 - x_4)$

Therefore, $P_{42} = 25 + 0.2\ (29 - 25) = 25 + 0.2(4) = 25 + 0.8 = 25.8$

$$P_{75} = \left(\frac{75\times10}{100}\right)^{th} \text{ item} = 7.5^{th} \text{ item} = 40$$

Note that $P_{75} = 40$. That is, 75% of the data values are less than P_{75} and the rest are above it.

Quartiles, deciles and percentiles for grouped data

You have just discussed quartiles, deciles and percentiles for ungrouped data. When we have a very large set of data, grouping the data in a frequency distribution will make it easier.

1 Quartiles

Example 17 Find the quartiles of the following grouped data.

Mark	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	41–45	46–50
f	1	2	17	25	11	13	18	5	4	4

Solution You need to first add the cumulative frequencies to the table.

Mark	1–5	6–10	11–15	16–20	21–25	26–30	31–35	36–40	41–45	46–50
f	1	2	17	25	11	13	18	5	4	4
cf	1	3	20	45	56	69	87	92	96	100

Q_1 is the 25^{th} item in the distribution. By assuming that the items are equally spread through each class, we calculate the value of the required item by means of proportions. Now since the first 20 items lie in earlier classes, Q_1 is the $25 – 20 = 5^{th}$ item in a class of 25 items. This means it lies $\left(\frac{5}{25}\right)^{th}$ of the way into the class. Since this class has an interval length of 5, $\left(\frac{5}{25}\right)^{th}$ of the way means that $\frac{5}{25}\times 5 = 1$ is to be added to the lower end. Now the quartile class starts at 16, so that the first quartile is 16+1=17.

Similarly, $Q_3 = 31 + \frac{75-69}{18}\times 5 = 32.67$. But, for a grouped data this approach may not be suitable. Thus, it will be good to look for a convenient way to finding quartiles.

Let us summarize the above example in the following formula:

The k^{th} **quartile** for a grouped frequency distribution is:

$$Q_k\,(k^{\text{th}}\text{quartile}) = B_L + \left(\frac{\frac{kn}{4} - cf_b}{f_k}\right)i$$

$k = 1, 2, 3$ and

B_L = lower class boundary of the k^{th} quartile class

cf_b = the cumulative frequency before the k^{th} quartile class

f_k = the number of observations (frequency) in the k^{th} quartile class

i = the size of the class interval

Steps to find quartiles for grouped data

1. Prepare a cumulative frequency distribution.
2. Find the class where the k^{th} quartile belongs: the $\left(\frac{kn}{4}\right)^{th}$ item.
3. Use the formula above.

Example 18 Find Q_1, Q_2 and Q_3 of the following distribution.

Ages	(f)	cum. fr
20 – 24	5	5
25 – 29	7	12
30 – 34	8	20
35 – 39	18	38
40 – 44	2	40

Solution $n = 40$,

Q_1 is $\left(\frac{40}{4}\right)^{th}$ item i.e. 10^{th} item which falls in the 2^{nd} class. $cf = 5, f_1 = 7$ and $i = 5$

$$Q_1 = 24.5 + \left(\frac{1 \times \frac{40}{4} - 5}{7}\right)5 = 24.5 + \frac{(10 - 5)5}{7} = 24.5 + \frac{5 \times 5}{7} = 24.5 + \frac{25}{7}$$

$Q_1 = 24.5 + 3.57 = 28.07$

Q_2 is $\left(\frac{2 \times 40}{4}\right)^{th}$ item $= 20^{th}$ item. Q_2 is found in the 3^{rd} class.

$$Q_2 = 29.5 + \left(\frac{\frac{2 \times 40}{4} - 12}{8}\right)5 = 29.5 + \left(\frac{20 - 12}{8}\right)5 = 29.5 + \left(\frac{8}{8}\right)5$$

$= 29.5 + 5 = 34.5$

Q_3 is $\left(\frac{3 \times 40}{4}\right)^{th}$ item $= 30^{th}$ item. It is found in the 4^{th} class.

$$Q_3 = 34.5 + \left(\frac{\frac{3 \times 40}{4} - 20}{18}\right)5 = 34.5 + \left(\frac{30 - 20}{18}\right)5 = 34.5 + \frac{10 \times 5}{18}$$

$Q_3 = 34.5 + 2.78 = 37.28$

Note:

Q_2 = median i.e. the 2^{nd} quartile is the same as the median.

Exercise 5.8

1 Find Q_1, Q_2, and Q_3 for each of the following data sets:

a 78, 68, 19, 35, 46, 58, 35, 35, 31, 10, 48, 28

b 1, 3, 5, 2, 8, 5, 6, 2, 3, 10, 7, 4, 9, 8

c

x	10	14	15	17	19	20	26
f	12	18	20	2	4	4	1

2 The following are quintals of fertilizer distributed to fifty farmers (You discussed this earlier).

24	19	26	28	29	25	32	22	24	18
32	13	31	26	18	18	26	14	24	24
28	32	23	16	24	19	34	31	13	36
16	23	32	41	34	24	31	23	18	42
6	8	24	26	34	18	32	19	28	14

a Find Q_1, Q_2, and Q_3.

b Find $Q_2 - Q_1$, $Q_3 - Q_2$ and $Q_3 - Q_1$. Write your conclusion.

3 Prepare a grouped frequency distribution, using 10 classes for the data in Question 2 and answer the following questions.

a Find Q_1, Q_2 and Q_3.

b Find the median and compare your result with Q_2.

4 Find Q_1, Q_2 and Q_3 of the following data. It is a distribution of marks obtained in a mathematics exam (out of 40).

Marks	10 – 14	15 – 19	20 – 29	30– 39
Number of students	7	12	8	9

a From the above data, if students in the top 25% are to be awarded a certificate, what is the minimum mark for a certificate?

b If students whose scores are in the bottom 25% of the marks are considered as failures, then what is the maximum failing mark?

2 Deciles

The j^{th} **decile** for grouped frequency distributions is calculated in a similar way as follows.

Steps to find deciles for grouped data

1. Find the class where the j^{th} decile belongs, which is the class that contains the $\left(\frac{jn}{10}\right)^{th}$ item.

2. Use the formula $D_j\left(j^{th} \text{ decile}\right) = B_L + \left(\frac{\frac{jn}{10} - cf_b}{f_c}\right)i, \quad j = 1, 2, 3, \ldots, 9.$

   Where B_L = Lower class boundary of the j^{th} – decile class.

   $n = \sum f$

   cf_b = cumulative frequency before the j^{th} – decile class.

   f_c = frequency of the j^{th} – decile class

   i = class size

Example 19 Find D_3 and D_7 of the following data.

weight	frequency	cum.fr.
40 – 49	6	6
50 – 59	10	16
60 – 69	17	33
70 – 79	3	36

Solution

a D_3 is $\left(\frac{3 \times 36}{10}\right)^{th}$ item = $(10.8)^{th}$ item. It is found in the 2^{nd} class.

So $D_3 = 49.5 + \frac{\left(\frac{3 \times 36}{10} - 6\right)10}{10} = 49.5 + 4.8 = 54.3.$

b $D_7 = \left(\frac{7 \times 36}{10}\right)^{th}$ item $= (25.2)^{th}$ item. It is in the 3^{rd} class

$D_7 = 59.5 + \left(\frac{\frac{7 \times 36}{10} - 16}{17}\right)10 = 59.5 + 5.41 = 64.91.$

3 Percentiles

The **j^{th} percentile** for grouped frequency distributions is calculated in a similar way as follows:

Steps to find percentiles for grouped data

1. Find the class where j^{th} percentile belongs, P_j is the $\left(\frac{jn}{100}\right)^{th}$ *item*
2. Use the following formula to find P_j

$$P_j = B_L + \left(\frac{\frac{jn}{100} - cf_b}{f_c}\right)i$$

Where B_L = Lower class boundary of the j^{th} percentile class.

$n = \sum f$

cf_b = cumulative frequency above the j^{th} percentile class.

f_c = frequency of the j^{th} – percentile class

i = size of class interval.

Example 20 Find P_{20} and P_{68} for the following frequency distribution.

weight	frequency	cum.fr.
40 – 49	6	6
50 – 59	10	16
60 – 69	17	33
70 – 79	3	36

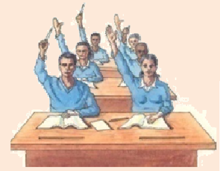
Solution: $P_{20} = \left(\frac{20 \times 36}{100}\right)^{th}$ item $= (7.2)^{th}$ item, which is in the 2^{nd} class.

So $P_{20} = 49.5 + \left(\frac{\frac{20 \times 36}{100} - 6}{10}\right)10 = 49.5 + 1.2 = 50.7$

P_{68} is $\left(\frac{68 \times 36}{100}\right)^{th}$ item $= 24.48^{th}$ item, which is in the 3^{rd} class.

So $P_{68} = 59.5 + \left(\frac{\frac{68 \times 36}{100} - 16}{17}\right)10 = 59.5 + 4.99 = 64.49$

ACTIVITY 5.4



1 From the above frequency distribution, find the median, 2nd quartile (Q_2), 5^{th} decile (D_5) and 50^{th} percentile (P_{50}). What do you observe? Did you see that median = $Q_2 = D_5 = P_{50}$?

Exercise 5.9

1 Find Q_2, Q_3, D_4, D_8, P_{12}, P_{24}, P_{87} for each of the following data sets:

a 78, 68, 19, 35, 46, 58, 35, 35, 31, 10, 48, 28

b

x	10	14	15	17	19	20	26
f	12	18	20	2	4	4	1

c

age	5 – 14	15 – 24	25 – 34	35 – 44	45 – 54
f	4	12	10	7	2

2 The daily profits in Birr of 100 shops are distributed in the following table. Find Q_1, Q_3 , D_4 and P_{70}.

Profit	1 – 100	101 – 200	201 – 300	301 – 400	401 – 500	501 – 600
No of shops	12	18	27	20	17	6

3 The following are quintals of fertilizer distributed to fifty farmers (You discussed this earlier).

24	19	26	28	29	25	32	22	24	18
32	13	31	26	18	18	26	14	24	24
28	32	23	16	24	19	34	31	13	36
16	23	32	41	34	24	31	23	18	42
6	8	24	26	34	18	32	19	28	14

a Find Q_1, Q_2, and Q_3.

b Find $Q_2 - Q_1$, $Q_3 - Q_2$ and $Q_3 - Q_1$. Write your conclusion.

4 Prepare a grouped frequency distribution, using 10 classes for the data in question

5 Answer the following questions.

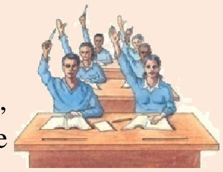
a Find Q_1, D_3, and P_{70}.

b Find the percentile of the farmers who received more than 20 quintals.

c If a farmer receives more than 75 percentile, find the minimum amount of quintals of fertilizer s/he receives.

5.1.4 Measures of Dispersion

ACTIVITY 5.5



For preparing a development plan of a farmers' association, researchers collected the following information on the yearly income of 20 farmers. Here are their incomes in Birr 1000.

10	15	20	12	13	20	8	9	10	6
12	13	8	14	5	6	8	20	12	6

a What is the mean yearly income of the farmers?

b Does the mean reflect the real living standard of each farmer?

c Before using the mean to reach to a conclusion, what other factors should be considered?

In Grade 9, you learned about the different measures of variation. In this section, we shall revise those concepts and see how to calculate them for grouped data.

Why do we need to study measures of variation?

Consider the following data: three copy typists A, B, C compete for a job. An exam is given for five consecutive days to measure their typing speed (words per minute).

A: 48, 52, 50, 45, 55 $\quad \bar{x}_A = 50$

B: 10, 90, 50, 41, 59 $\quad \bar{x}_B = 50$

C: 50, 50, 50, 50, 50 $\quad \bar{x}_C = 50$

The average (mean) speed of all three is the same (50 words per minute). Which typist should be selected? The next criterion should be consistency.

Definition 5.9

The degree to which numerical data is spread about an average value is called the **variation** or **dispersion** of the data.

The common measures of variation that we are going to see are **Range**, **Variance** and **Standard Deviation**.

Range

Range is the difference between the maximum and the minimum values in a data set.

$$\text{Range} = x_{\max} - x_{\min}$$

Example 21 Find the range of

a 4, 6, 2, 10, 18, 25

b

x	2	5	7	8	10
f	3	4	9	2	6

Solution:

a $x_{max} = 25, x_{min} = 2$; Range $= x_{max} - x_{min} = 25 - 2 = 23$

b Range = 10 – 2 = 8

Range for grouped data

Definition 5.10

Range for grouped data is defined as the difference between upper class boundary of the highest class $B_u\,(H)$ and the lower class boundary of the lowest class $B_L(L)$. That is,

$$R = B_u(H) - B_L(L)$$

Example 22 Consider the following data, what is the range of this distribution?

x	5 – 10	11 – 16	17 – 22
f	4	9	6

Solution: From the grouped frequency distribution, the range is calculated using

$B_u(H) = 22.5$, $B_L(L) = 4.5$

$\therefore$ R = 22.5 – 4.5 = 18

Advantages and limitations of range

Advantage of Range

✓ It is simple to compute

Limitation of Range

✓ It only depends on extreme values.

✓ It doesn't consider variations of values in between.

✓ It is highly affected by extreme values.

Variance and standard deviation

The standard deviation is the most commonly used measure of dispersion. The value of the standard deviation tells how closely the values of a data set are clustered around the mean. In general, a lower value of the standard deviation for a data set indicates that the values of the data set are spread over a relatively small range around the mean. On the other hand, a large value of the standard deviation for a data set indicates that the values of that data set are spread over a relatively large range around the mean.

Definition 5.11

Variance is the average of the squared deviation of each item from the mean.

Variance for ungrouped data

If $x_1, x_2, x_3, ..., x_n$ are n observed values, then variance for the sample data is given by

$$\text{Variance } (s^2) = \frac{(x_1 - \bar{x})^2 + ... + (x_n - \bar{x})^2}{n} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

where, $\bar{x}$ = mean

s^2 = variance.

n = number of values

✍Note:

The quantities $x - \bar{x}$ in the above formula are the deviations of x from the mean.

Definition 5.12

The positive square root of variance is called **standard deviation**.

Standard deviation $(sd) = \sqrt{\text{variance}}$

$$sd = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x}^2)}{n}}$$

Steps to calculate variance for ungrouped data

a Calculate the mean of the distribution.

b Find the deviation of each value from the mean and square it.

c Add the squared deviations.

d Divide the sum obtained in step 3 by n.

Example 23 Find the variance and standard deviation of the following data.

20, 16, 12, 8, 18, 5, 9, 24

Solution: $\bar{x} = \frac{20 + 16 + 12 + 8 + 18 + 5 + 9 + 24}{8} = 14$

X	$x - \bar{x}$	$(x - \bar{x})^2$
20	6	36
16	2	4
12	–2	4
8	–6	36
18	4	16
5	–9	81
9	–5	25
24	10	100

$\sum (x - \bar{x})^2 = 302$

Variance $s^2 = \frac{\sum (x - \bar{x}^2)}{n} = \frac{302}{8} = 37.75$

Standard deviation $s = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \sqrt{37.75} = 6.14$

If $x_1, x_2, \ldots, x_n$, are values with corresponding frequencies $f_1, f_2, \ldots, f_n$, the variance is given by

$$s^2 = \frac{f_1(x_1 - \bar{x})^2 + f_2(x_2 - \bar{x})^2 + \ldots + f_n(x_n - \bar{x})^2}{\sum f_i} = \frac{\sum_{i=1}^{n} f_i(x_i - \bar{x})^2}{\sum_{i=1}^{n} f_i}$$

Steps to calculate variance from frequency distributions

a Find the mean of the distribution.

b Find the deviation of each item from the mean and square it.

c Multiply the squared deviations by their corresponding frequencies and add.

d Divide the sum by $\sum f_i$.

Example 24 Find the variance and standard deviation of the following data.

x	F	$(x - \bar{x})$	$(x - \bar{x})^2$	$f(x - \bar{x})^2$
2	3	– 4.88	23.8	71.44
5	4	– 1.88	3.53	14.14
7	9	0.12	0.0144	0.1296
8	2	1.12	1.254	2.5088
10	6	3.12	9.73	58.41
	24		$\sum f(x - \bar{x})^2$	= 146.63

Solution: $\bar{x} = \frac{165}{24} = 6.88$

variance, $s^2 = \frac{\sum f(x - \bar{x})^2}{n} = \frac{146.63}{24} = 6.11$

Standard deviation = $\sqrt{\text{variance}} = \sqrt{6.11} = 2.47$

Variance for grouped data

✍Note:

In a grouped frequency distribution, every class is represented by its class mark or class midpoint.

The variance for grouped data is given by

$$s^2 = \frac{\sum f_i(x_i - \bar{x})^2}{\sum f_i}$$ where x_i is the midpoint of each class (class mark).

Steps to find variance from a grouped frequency distribution

a Find the class mark for each class.

b Find the mean of the grouped data.

c Find the deviation of each class mark from the mean and square it.

d Find the sum of the squared deviations.

e Divide the sum obtained in step d by $\sum f_i$.

Example 25 Find the variance and standard deviation of the following distribution.

age (x)	frequency (f)	class mark (x_i)	fx_i	$x_i - \bar{x}$ ($x_i - 7$)	$(x_i - \bar{x})^2$ $(x_i - 7)^2$	$f(x_i - \bar{x})^2$ $f(x_i - 7)^2$
0 – 4	4	2	8	–5	25	100
5 – 9	8	7	56	0	0	0
10 – 14	2	12	24	5	25	50
15 – 19	1	17	17	10	100	100

$\sum f_i = 15$ $\quad\quad \sum f_i x_i = 105$ $\quad\quad \sum f_i(x_i - \bar{x})^2 = 250$

Solution:

$$\text{mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{105}{15} = 7$$

$$\text{Variance} = \frac{\sum f_i(x_i - \bar{x})^2}{\sum f_i} = \frac{250}{15} = 16.67$$

$$\text{Standard deviation} = \sqrt{\frac{\sum f_i(x_i - \bar{x})^2}{\sum f_i}} = \sqrt{\frac{250}{15}} = 4.08$$

Merits and Demerits of standard deviation

Merits

1 It is rigidly defined.

2 It is based on all observations.

Demerits

1 The process of squaring deviations and then taking the square root of their mean is complicated.

2 It attaches great weight to extreme values, as squared deviations are used.

Exercise 5.10

1 Find the range, variance and standard deviation for each of the following data.

a 18, 2, 4, 6, 10, 7, 9, 11 b 3, 4, 5, 5, 6, 7, 7, 7

c

x	31	35	36	40	42	50
f	7	8	2	12	6	3

d

Class	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79	80 – 89
Frequency	8	10	16	14	10	12

2 Why do we study measures of variation?

3 If the standard deviation of $x_1, x_2, x_3, \ldots, x_n$ is 3, then what is the standard deviation of $2x_1 + 3, 2x_2 + 3, \ldots, 2x_n + 3$?

4 The standard deviation of the temperature for one week in a certain city is zero. What can you say about the temperature of that week?

5 Two basketball players scored points for their team. The scores were recorded for 9 games as follows:

Player A	3	4	5	6	7	8	9	10	11
Player B	4	3	5	6	7	8	9	9	1

a Calculate the standard deviation of the points of each player.

b Which player, A or B, is more consistent in scoring points for his team? How do you know?

6 Consider the following raw data representing yield of Barley (in quintals) of three farmers from their respective hectare of land for consecutive 8 years.

Farmer 1	12	14	11	13	17	18	12	13	11
Farmer 2	14	13	15	13	14	13	15	13	13
Farmer 3	12	5	14	3	17	8	4	12	13

a Determine the range, variance and standard deviation of each of the three farmers.

b Who of the farmers has higher variation in yield? What does this tell?

c Who of the farmers has lesser variation in yield?

d Who of the farmers has consistent yield?

Group Work 5.3

Do the following in groups. Apply as many of the formulae as necessary.

1 Design and carry out a questionnaire survey to find out how students in your school spend their spare time. You need to find out:

a the average hours they spend on entertainment (watching TV, games, etc);

b the average hours they spend on chores (to help their family, to earn money, etc);

c the average hours they spend on study;

d the average mark obtained at the end of the year.

e Can you conclude anything about the effect of the way they use their spare time on their academic performance?

2 Investigate how students come to school, by taking a sample. Do they come by bus, car, on foot, cycle or any other means? How does this relate to family income, distance of school from home, gender, etc?

3 Take a sample of students and measure and record their heights, weights and ages. Consider questions like whether or not their heights are as expected for their age groups. You could take their gender and weight into consideration.

5.2 PROBABILITY

In Grade 9, you have studied basic concepts of probability. In this section you will revise some definitions before we proceed to the next section.

1. An **Experiment** is an activity (measurement or observation) that generates results (outcomes).
2. An **Outcome** (Sample point) is any result obtained in an experiment.
3. A **Sample Space** (S) is a set that contains all possible outcomes of an experiment.
4. An **Event** is any subset of a sample space.

Example 1 When a "fair" coin is tossed, the possible results are either head (H) or tail (T). Consider an experiment of tossing a fair coin twice.

- **a** What are the possible outcomes?
- **b** Give the sample space.
- **c** Give the event of H appearing on the second throw.
- **d** Give the event of at least one T appearing.

Solution:

- **a** HH, HT, TH, TT
- **b** S = {HH, HT, TH, TT}
- **c** A = {HH, TH}
- **d** B = {HT, TH, TT}

✍Note:

In tossing a coin, if the coin is fair, the two possible outcomes have an equal chance of occurring. In this case, we say that the outcomes are **equally likely**.

Probability of an event (E)

If an event E can happen in m ways out of n equally likely possibilities, the probability of the occurrence of an event E is given by

$$P(E) = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} = \frac{n(E)}{n(S)} = \frac{m}{n}$$

Example 2 A box contains 4 red and 5 black balls. If one ball is drawn at random, what is the probability of getting a

- **a** red ball?
- **b** black ball?

Solution Let event R = a red ball appears and event B = a black ball appears. Then,

a $P(R) = \dfrac{n(R)}{n(S)} = \dfrac{4}{9}$ **b** $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{5}{9}$

Example 3 If a number is to be selected at random from the integers 1 through to10, what is the probability that the number is

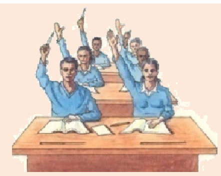
a odd? **b** divisible by 3?

Solution $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

a odd is the event $E = \{1, 3, 5, 7, 9\} \Rightarrow P(\text{odd}) = \dfrac{\text{number of odds}}{\text{total numbers}} = \dfrac{5}{10} = \dfrac{1}{2}$

b divisible by 3 is the event $E = \{3, 6, 9\} \Rightarrow P(\text{divisible by 3}) = \dfrac{3}{10}$

ACTIVITY 5.6



A fair die is tossed. What is the probability of getting

a the number 4? **b** an even number?

c the number 7? **d** either 1, 2, 3, 4, 5 or 6?

e a number different from 5?

5.2.1 Permutation and Combination

In the previous example of tossing a fair coin twice, the number of all possible outcomes was only four. To find the probability of the event A = {HH, TH}, you have to count the number of outcomes in event A (which is 2) and divide by n (S). Thus, we have

$$P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$$

Now, if the experiment is tossing a coin five times, what is the total number of possible outcomes? If an event E is defined by "3 heads and 2 tails", then how do you find $n(E)$?

From this, you can observe that counting plays a very important role in finding probabilities of events.

In this section, you shall see some mathematical techniques which will help you to simplify counting problems. When the number of possible outcomes is very large, it will be difficult to find the number of possible outcomes by listing. So you have to investigate different counting techniques which will help you to find the number of elements in an event and a possibility set.

Fundamental principles of counting

There are two fundamental principles that are helpful for counting. These are the multiplication principle and the addition principle.

Multiplication principle

Before we state the principle, let us consider the following example.

Example 4 Suppose Nuria wants to go from Harrar via Dire Dawa to Addis Ababa. There are two minibuses from Harrar to Dire Dawa and 3 buses from Dire Dawa to Addis Ababa. How many ways are there for Nuria to travel from Harrar to Addis Ababa?

Solution: Let M stand for Minibus and B stand for Bus.

There are $(2 \times 3) = 6$ possible ways.

These are M_1B_1, M_1B_2, M_1B_3, M_2B_1, M_2B_2 , M_2B_3.

The example above illustrates the **Multiplication Principle of Counting**.

If an event can occur in m different ways, and for every such choice another event can occur in n different ways, then both the events can occur in the given order in $m \times n$ different ways. That is, the number of ways in which a series of successive things can occur is found by multiplying the number of ways each thing can occur.

In the above illustration, Nuria has two possible choices to go from Harar to Dire Dawa and three alternatives from Dire Dawa to Addis Ababa.

The total number of ways is $2 \times 3 = 6$.

Example 5 Suppose there are 5 seats arranged in a row. In how many different ways can five people be seated on them?

Solution: The first man has 5 choices, the 2^{nd} man has 4 choices, the 3^{rd} man has 3 choices, the 4^{th} has two choices, and the 5^{th} has only one choice. Therefore, the total number of possible seating arrangements is

$$5 \times 4 \times 3 \times 2 \times 1 = 120.$$

Example 6 Suppose that you have 3 coats, 8 shirts and 6 different trousers. In how many different ways can you dress?

Solution: $3 \times 8 \times 6 = 144$ ways.

Addition principle

If an event E_1 can occur in m ways and another event E_2 can happen in n ways, then either of the events can occur in $n + m$ ways. This is true when E_1 and E_2 are mutually exclusive events.

✍Note:

Two events are said to be mutually exclusive, if both cannot occur simultaneously.

In tossing a coin, Head and Tail are mutually exclusive events because they cannot appear at the same time.

Example 7 A question paper has two parts where one part contains 4 questions and the other 3 questions. If a student has to choose only one question, from either part, in how many ways can the student do it?

Solution: The student can choose one question in $4 + 3 = 7$ ways.

Combined counting principles

The fundamental counting principles can be extended to any number of sequences of events.

Example 8 A question paper has three parts: language, arithmetic and aptitude tests. The language part has 3 questions, the arithmetic part has 6 questions and the aptitude part has 5 questions. If a student is expected to answer one question from each of two of the three parts, with arithmetic being compulsory, in how many ways can the student take the examination?

Solution: The student can either take language and arithmetic or arithmetic and aptitude. This gives $3 \times 6 + 5 \times 6 = 48$ possibilities.

Exercise 5.11

1 In an experiment of selecting a number from 1 – 10, which of the following cannot be an event?

- **a** The number is "even and prime".
- **b** The number is "even and multiple of 5".
- **c** The number is multiple of 3.
- **d** The number is zero.

2 In an experiment of tossing three coins at a time,

- **a** Determine the sample space.
- **b** Find the probability of getting two heads.

3 A box contains 2 Red and 3 Black balls. If two balls are drawn at random,

a Determine the possible outcomes

b Find the probability of getting 2 Red balls.

c Find the probability of getting 1 Red and 1 Black balls.

4 Suppose you have six different books. In how many different ways can you arrange these books on a shelf?

5 There are three gates to enter a school and two doors to go into a classroom. In how many different ways can a student get into a class from outside?

6 In a classroom there are 50 students from whom 27 are female students. If one student is selected at random, what is the probability of getting male student?

Example 9 Suppose there are only three seats and there are five people to be seated. In how many ways can these people be seated on the three seats?

Definition 5.13

For any positive integer n, n factorial denoted as $n!$ is defined as

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$$

We define $0! = 1$.

Example 10 Calculate

a $3!$ **b** $5!$ **c** $\frac{8!}{4!}$

Solution:

a $3! = 3 \times 2 \times 1 = 6$ **b** $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

c $\frac{8!}{4!} = \frac{8 \times 7 \times 6 \times 5 \times 4!}{4!} = 8 \times 7 \times 6 \times 5 = 1680$

Permutation

Definition 5.14

A **Permutation** is the number of arrangements of objects with attention given to the order of arrangements.

In Example 5 above, the 5 people can be arranged in 5 seats in

$5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

The number of permutations of a set of n objects taken all together is denoted by $P(n, n)$ or ${}_nP_n$ and is equal to $n!$

Thus, $P(n, n) = n!$

Example 11

a Give all the permutations of three letters A, B and C.

b Suppose we have 5 people to be seated in only 3 seats. In how many ways can they sit?

Solution:

a The three letters A, B and C can be arranged in

$P(3, 3) = 3! = 3 \times 2 \times 1 = 6$ different permutations.

These are: ABC, ACB, BAC, BCA, CAB and CBA.

b The first chair can be filled by any one of the 5 people, the second by any one of the remaining 4 people and the third by any of the remaining 3 people. By the multiplication principle, this gives $5 \times 4 \times 3 = 60$ possibilities.

$$60 = 5 \times 4 \times 3 = \frac{5\times4\times3\times2\times1}{2\times1} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$$

Definition 5.14

The number of permutations of n objects taken r at a time, where $0 < r \le n$, is denoted by $P(n, r)$ or ${}_nP_r$ and is given by $P(n,r) = \frac{n!}{(n-r)!}$.

Group Work 5.4

Do the following in groups

1 Compute the following.

a ${}_6P_2$ **b** ${}_8P_5$ **c** ${}_{1000}P_{999}$

2 Five students are contesting an election for 5 places on the committee of the environmental protection club in their school. In how many ways can their names be listed on the ballot paper?

3 From the letters A, B, C, D, E, how many three – letter "words" can be formed? (*the words need not have meanings*)

4 Consider the word **CALL.** If you think of the two Ls as different, say L_1 and L_2, then CL_1AL_2 and CL_2AL_1 would have been different. But, as it happens, L_1 and L_2 represent the same letter L. Taking this into consideration, find all the 12 (distinct) permutations of **CALL**.

Permutation of duplicate items

If there are n objects with n_1 alike objects of a first type, n_2 alike objects of a second type,…, and n_r alike objects of the r^{th} type, where $n_1 + n_2 + \ldots + n_r = n$, then there are $\frac{n!}{n_1!n_2!\cdots n_r!}$ permutations of the given n objects.

For the above group work, in the word CALL, the number of permutations will be

$$\frac{4!}{2!} = 12$$

Exercise 5.12

1 Find the Factorial of each of the following numbers

a 6 **b** 8 **c** 12

2 How many four – digit numbers can be formed from the digits 1, 3, 5, 7, 8 and 9 where a digit is used at most once?

a if the numbers must be even? **b** if the numbers are less than 3000?

3 Two men and a woman are lined up to have their picture taken. If they are arranged at random, find the number of ways that

a the woman will be on the left in the picture.

b the woman will be in the middle of the picture.

4 Find the number of permutations that can be made out of the letters of the word "MATHEMATICS". In how many of these permutations

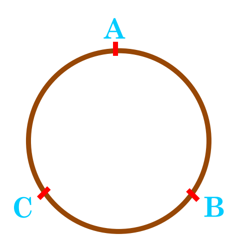
a do the words start with C? **b** do all the vowels occur together?

c do the words begin with H and end with S?

5 In a library there are 3 Mathematics, 4 Geography and 3 Economics books. If each of them will be put on a shelf and each type of a book are identical, in how many ways can these books be arranged?

6 Verify that ${}_nP_{n-1} = {}_nP_n$.

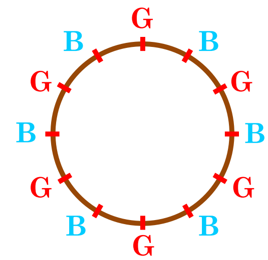
Circular permutations



Is there a difference between arrangements of objects in a straight line and around a circle? Consider three letters A, B, C and try to find the number of different permutations along a circle. Since it is difficult to indicate the relative position of objects in a circle, we fix the position of one object and arrange the remaining objects.

If n objects are to be arranged on a circle (along the circumference of a circle), then the number of circular permutations is given by $(n - 1)!$

Example 12



a 7 people are to sit around a circular table. In how many different ways can these people be seated?

b In how many ways can 6 boys and 6 girls sit around a table of 12 seats, if no two girls are to sit together?

Solution

a The number of ways these 7 people sit around a round table is

$(7 - 1)! = 6! = 720$ ways.

b First allot seats to the boys, as shown in the diagram.

Now the 6 boys can sit in $(6 - 1)! = 5! = 120$ ways.

Next the 6 girls can occupy seats marked (G). There are 6 such seats. This can be done in ${}_6P_5 = 720$ ways. By the Fundamental Principle of Counting, the required number of ways is

$120 \times 720 = 86{,}400$ ways.

Combination

Before you define the concept of combinations, see the following example that helps to illustrate how it is different from permutations.

Three students A, B and C volunteer to serve on a committee. How many different committees can be formed containing two students?

Let us try to use permutations of two out of three: ${}_3P_2 = \dfrac{3!}{(3-2)!} = 6$. The possible arrangements are AB, AC, BC, BA, CA, CB. But AB and BA, AC and CA, BC and CB contain the same members. Hence AB and BA cannot be considered as different committees, because the order of the members does not change the committee.

Thus, the required number of possible committee members is not six but three: AB, AC and BC. This example leads us to the definition of combinations.

> **Definition 5.15**
>
> The number of ways r objects can be chosen from a set of n objects without considering the order of selection is called the number of **combinations** of n objects taking r of them at a time, denoted by
>
> $$C(n,r) = \binom{n}{r} = C_r^n \text{ and defined by } C(n,r) = \frac{n!}{(n-r)!r!}, 0 < r \le n$$

To arrive at a formula for ${}_nC_r$, observe that the r objects in ${}_nP_r$ can be arranged among themselves in $r!$ ways.

$$\text{Hence,}\quad C(n,r)=\frac{{}_nP_r}{r!}=\frac{\frac{n!}{(n-r)!}}{r!}=\frac{n!}{(n-r)!r!}$$

Therefore, the number of possible combinations of n objects taken r at a time is given by the formula

$$\binom{n}{r}=C(n,r)=\frac{n!}{(n-r)!r!}\ ,\ 0<r\leq n$$

From this, you can see that the number of ways that a committee of two members can be selected from three individuals is given by

$$C\,(3,2)\ =\ \frac{3!}{1!2!}=3\text{ ways.}$$

Example 13 Compute the following.

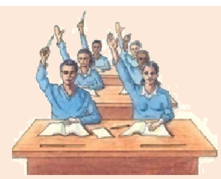
a $C\,(6,2)$ **b** $C\,(10,4)$

Solution:

a $C\,(6,2)=\frac{6!}{(6-2)!2!}=\frac{6!}{4!2!}=\frac{6\times5\times4!}{4!\times2\times1}=15$

b $C\,(10,4)=\frac{10!}{6!4!}=210$

ACTIVITY 5.7



Show each of the following.

a $C(n,0)=1$ **b** $C(n,r)=C(n,n-r)$

c $\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$

Example 14

a In an examination paper, there are 12 questions. In how many different ways can a student choose eight questions in all, if two questions are compulsory?

b In how many different ways can three men and three women be selected from six men and eight women?

c In how many ways can Bekele invite at least one of his friends out of 5 friends to an art exhibition?

d A committee of 7 students has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee contains

i exactly three girls? **ii** at least three girls?

iii 2 girls and 5 boys?

Solution

a Since 2 questions are compulsory, the student is left with a choice of selecting 6 questions from the remaining 10 questions.

Hence, he/she can do it in $C(10, 6)$ ways i.e. $C(10, 6) = \frac{10!}{4!6!} = 210$ ways.

b Three men from six can be selected in $\binom{6}{3}$ ways. Three women from 8 can be selected in $\binom{8}{3}$ ways. Therefore, the total number of ways that a committee of three men and three women be selected out of 6 men and 8 women is given by

$$\binom{6}{3}\times\binom{8}{3} = 20\times56 = 1120 \text{ ways (by the Multiplication principle).}$$

c At least one means that he can invite either one, two, three, four or five.

Therefore, the total number of ways in which he can invite at least one of his friends is given by (Addition principle)

$$C(5,1) + C(5,2) + C(5,3) + C(5,4) + C(5,5) = 5 + 10 + 10 + 5 + 1 = 31.$$

d **i** When exactly 3 girls are included in the committee, the remaining members will be 4 boys.

∴ The total number of ways of forming a committee is

$C(4, 3)\times C(9, 4) = 4 \times 126 = 504$ ways.

ii At least 3 girls are included means the committee will consist of either 3 girls and 4 boys or 4 girls and 3 boys.

∴ Total number of ways of forming a committee is given by

$$[C(4,3)\times C(9,4)] + [C(4,4)\times C(9,3)] = 4 \times 126 + 1 \times 84$$
$$= 504 + 84 = 588 \text{ ways.}$$

iii Two girls and 5 boys can be selected in

$C(4,2)\times C(9,5) = 6\times 126 = 756$ ways.

Exercise 5.13

1 Compute each of the following.

a C (8, 0) **b** C (n, n) **c** C (8, 6)

2 If C $(n, 6) = C(n, 4)$, find n.

3 In how many ways can a committee of 5 be selected from 10 people willing to serve?

4 A committee of 5 students has to be formed from 9 boys and 9 girls. In how many ways can this be done when the committee consists of

a 2 girls and 3 boys? **b** all boys? **c** all girls?

d at least 3 boys? **e** at most 3 girls?

5 In Ethiopia there are 20 Premier league Soccer teams.

a In one round how many games are there?

b If five of the teams represent one company, find the number of ways pairs of teams representing different companies can play a game.

6 In a box there are 3 red, 4 white and 5 black balls. If we choose three balls at random, what is the number of ways such that:

a one ball is white? **b** 3 of them are black? **c** at most 2 are red?

5.2.2 Binomial Theorem

Group Work 5.5

Do the following in groups:

1 For any $1 \le n \le 5$, expand $(a + b)^n$.

2 Generalize the formula for any natural number n.

3 Answer the following from what you have done in 2 above :

a How many terms are there?

b What is the pattern you notice concerning the exponents of "a" ? What about the exponents of "b"?

c Given a term, what is the sum of the exponents of a and b?

d Give the coefficients of the first and the last terms.

e Can you express the coefficients using combination notation?

f Complete the "Pascal's triangle" given below.

1 . . . . Coefficient in $(a + b)^0$

1 1 . . . . Coefficients in $(a + b)^1$

1 2 1 . . . . Coefficients in $(a + b)^2$

__ __ __ __ . . . . Coefficients in $(a + b)^3$

__ __ __ __ __. . . . . Coefficients in $(a + b)^4$

__ __ __ __ __ __. . . . . Coefficients in $(a + b)^5$

g Consider the terms in the middle. How is a term there related to the two terms immediately above it?

h How does your observation relate to Activity 5.7 c?

ACTIVITY 5.8

Using Pascal's Triangle, expand $(a+b)^6$, $(a+b)^7$ and $(a+b)^8$.

Binomial theorem

For a non – negative integer n, the binomial expansion of $(x+y)^n$ is given by

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \ldots + C(n,r)x^{n-r}y^r + \ldots + C(n,n)y^n$$

Example 15 Expand $(x+y)^4$.

Solution: $(x+y)^4 = C(4,0)x^4 + C(4,1)x^3y + C(4,2)x^2y^2 + C(4,3)xy^3 + C(4,4)y^4$

$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$

Exemple 16 Find the coefficient of x^2y^3 in the expansion of $(x+y)^5$.

Solution : $(x+y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5.$

Thus, the coefficient of x^2y^3 is $\binom{5}{3} = \frac{5!}{3!2!} = \frac{5\times4}{2} = 10.$

Exercise 5.14

1 Expand each of the following using the Binomial Theorem:

a $(a+b)^5$ **b** $(a+b)^7$ **c** $(3x-4y)^6$

2 Without writing all the expanded terms, answer the following

a What is the coefficient of a^3b^5 in the expansion of $(a+b)^8$?

b What is the coefficient of a^4b^2 in the expansion of $(a+b)^6$?

c What is the coefficient of the term containing a^2b^4 in $(a+b)^6$?

3 In expanding $(x+y)^3$ find the terms that have equal coefficients.

4 In the expansion of $(a+b)^{10}$

a How many terms are there?

b Find the terms whose coefficient is 45.

5 In the expansion of $(2x+5y)^5$

a What is the coefficient of the term x^2y^3?

b Find the terms whose coefficient is 400.

6 Find the constant term in the expansion of $\left(x+\frac{3}{x^3}\right)^4$

5.2.3 Random Experiments and Their Outcomes

At the beginning of this section, you saw the basic definitions of experiment, event and sample space. In this section, you will use these terms again and also see additional concepts.

Definition 5.16

A **random experiment** is an experiment (activity) which produces some well defined results. If the experiment is repeated under identical conditions it does not necessarily produce the same results.

Example 17 Give the outcomes for each of the following experiments.

a Tossing a coin **b** Tossing a pair of coins

c Rolling a die **d** Rolling a pair of dice

Solution:

a {H, T} **b** {HH, HT, TH, TT} **c** {1, 2, 3, 4, 5, 6}

d

(1 ,1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

✍**Note:**

Outcomes of a random experiment are said to be equally likely when there is no reason to expect any one of the outcomes in preference to another. That is, each element has equal chance of being chosen.

Example 16 If a fair die is thrown, any one of the outcomes 1, 2, 3, 4, 5, 6 has an equal chance of appearing at the top. Therefore, they are considered as equally likely.

✍**Note:**

In a random experiment, the outcomes which insure the happening of a particular result are said to be favourable outcomes to that particular result.

Example 18

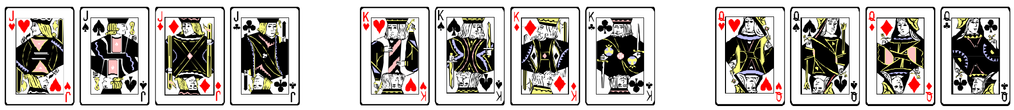
a A fair die is thrown. How many favourable outcomes are there for getting an even number?

b In picking a playing card from a pack of 52 cards, what is the number of favourable outcomes to getting a picture card?

Solution:

a There are 3 favourable outcomes. These are 2, 4 and 6.

b There are 12 favourable outcomes - 4 Jacks, 4 Queens and 4 Kings.



Four Jacks **Four Queens** **Four Kings**

Figure 5.1

5.2.4 Events

Recall that any subset of a sample space is called an **event** and is usually denoted by E. An event is a collection of sample points.

Example 19 The four faces of a regular tetrahedron are numbered 1, 2, 3 and 4. If it is thrown and the number on the bottom face (on which it stands) is registered, then list the events of this experiment.

Solution The sample space = {1, 2, 3, 4}.

The possible events are {1}, {2}, {3} and {4}.

ACTIVITY 5.9

List some events of the following experiments.

a Tossing a coin three times.

b Inspecting produced items.

c Selecting a number at random from integers 1 through to 12.

d Drawing a ball from a bag containing 4 red and 6 white balls.

e A married couple expecting a child.

Types of events

a **Simple Event (Elementary Event)** is an event containing exactly one sample point.

Example 20 In a toss of one coin, the occurrence of tail is a simple event.

b **Compound Event** When two or more events occur simultaneously, their joint occurrence is known as a compound event, an event that has more than one sample point.

Example 21 When a die is rolled, if you are interested in the event "getting even number", then the event will be a compound event, i.e. { 2, 4, 6}.

We can determine the possible number of events that can be associated with an experiment whose sample space is S. As events are subsets of a sample space, and any set with m elements has 2^m subsets, the number of events associated with a sample space with m elements is 2^m. (Sometimes this is called the *exhaustive number of events*).

Example 22 Suppose our experiment is tossing a fair coin. The sample space for this experiment is S = {H, T}. Thus, this sample space has a total of four possible events that are subsets of S. The list of the possible events is { }, {H}, {T}, and {H, T}.

Occurrence or Non-occurrence of an event

During a certain experiment, there are two possibilities associated with an event, namely, occurrence or non-occurrence of the event.

Example 23 If a die is thrown, then S = {1, 2, 3, 4, 5, 6}. Let E be the event of getting odd number, then $E = \{1, 3, 5\}$. When we throw the die, if the outcome is 3, as $3 \in E$, then we say that E has occurred. If in another trial, the outcome is 4, then as $4 \notin E$, we say that E has not occurred (not E).

c **Complement of an Event** E, denoted by E' (not E) consists of all events in the sample space that are not in E.

Example 24 Let a die be rolled once. Let E be the event of a prime number appearing at the top i.e. $E = \{2, 3, 5\}$. Give the complement of the event.

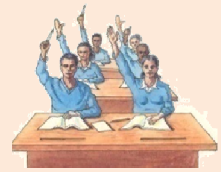
Solution: $E' = \{1, 4, 6\}$.

✍Note:

$E' = S - E = \{w: w \in S \text{ and } w \notin E\}$

Algebra of events

ACTIVITY 5.10



Discuss the following:

a Union and intersection of two events:

b State properties of union and intersection.

c What are exhaustive and mutually exclusive events?

d When are two events called independent?

Note:

Since events are sets (subsets of the sample space) one can form union, intersection and complement of them. The operations obey algebra of sets, like commutativity, distributivity, De Morgan's laws and so on.

d **Exhaustive Events** are events where at least one of them must necessarily occur every time the experiment is performed.

Example 25 If a die is thrown give instances of exhaustive events.

Solution: The sample space is S = {1, 2, 3, 4, 5, 6}. From this, the events {1}, {2}, {3}, {4}, {5}, {6} are exhaustive events. The events {1, 2}, {3, 4}, {4, 5, 6} are also exhaustive events for this experiment.

More generally, events E_1, E_2, …, E_n form a set of exhaustive events of a sample space S where $E_1, E_2, \ldots, E_n$ are subsets of S and $E_1 \cup E_2 \cup \ldots \cup E_n = S$.

e **Mutually Exclusive Events** are events that cannot happen at the same time.

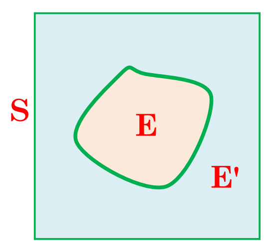


Figure 5.2

Example 26 Say whether or not the following are mutually exclusive events.

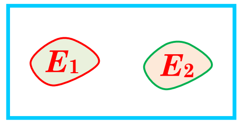
i When a coin is tossed once, the events {H} and {T}.

ii When a die is rolled, E_1 = getting an even number

E_2 = getting a prime number

Solution:

i Either we get head or tail but we cannot get both at the same time. Thus, {H} and {T} are mutually exclusive events.



$E_1 \cap E_2 = \varnothing$

ii E_1 and E_2 are not mutually exclusive because 2 is even and prime at the same time.

f **Exhaustive and Mutually Exclusive Events:** If S is a sample space associated with a random experiment and if $E_1, E_2, ..., E_n$ are subsets of S such that

i $E_i \cap E_j = \varnothing$ for $i \neq j$ and,

ii $E_1 \cup E_2 \cup ... \cup E_n = S$, then the collection of the events $E_1, E_2, ..., E_n$ forms a mutually exclusive and exhaustive set of events.

Example 27 If a die is thrown, the events {1}, {2}, {3}, {4}, {5}, {6} are mutually exclusive and exhaustive events. But, the events {1, 2}, {3, 4}, {4, 5, 6} are not because $\{3, 4\} \cap \{4, 5, 6\} \neq \varnothing$.

g **Independent Events:** Two events are said to be independent, if the occurrence or non occurrence of one event does not affect the occurrence or non-occurrence of the other.

Example 28 In a simultaneous throw of two coins, the event of getting a tail on the first coin and the event of getting a tail on the second coin are independent.

Example 29 If a card is drawn from a well shuffled pack of cards and is replaced before drawing a second card, then the result from drawing the second card is independent of the result of the first drawn card.

h **Dependent Events** Two events are said to be dependent, if the occurrence or non occurrence of one event affects the occurrence or non-occurrence of the other.

Example 30 If a card is drawn from a well shuffled pack of cards and the card is not replaced, then the result of drawing a second card is dependent on the first draw.

5.2.5 Probability of an Event

In grade 9, you dealt with an experimental approach to probability. You also discussed the definition of theoretical probability of an event. Probability can be measured by three different approaches.

- **a** The classical (mathematical) approach.
- **b** The empirical (relative frequency) approach.
- **c** The axiomatic approach.

a The classical approach

This is the kind of probability that you discussed in grade 9. In the classical approach to probability, if all the outcomes of a random experiment are equally likely and mutually exclusive, then the probability of an event E is

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{number of outcomes favoring } E}{\text{number of all possible outcomes}}$$

Example 31 A fair die is tossed once. What is the probability that an even number appears?

Solution: E = an even number shows up = $\{2, 4, 6\}$. Then, $P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$.

b The empirical approach

This approach is based on the relative frequency of an event (or outcome) when an experiment is repeated a large number of times. Here, the probability of an event E is the proportion of outcomes favourable to E in the experiment.

Thus, $P(E) = \frac{\text{frequency of } E}{\text{total number of observations}} = \frac{f_E}{N}$

Example 32 If records show that 60 out of 100,000 bulbs produced are defective (D), then the probability of a newly produced bulb being defective is given by

$$P(D) = \frac{f_D}{N} = \frac{60}{100{,}000} = 0.0006$$

c The axiomatic approach

In this approach, the probability of an event is given as a function that satisfies the following definition:

Let S be the sample space of a random experiment. With each event E we associate a real number called **the probability of E**, denoted by $P(E)$, that satisfies the following properties called **axioms** (or **postulates**) **of probability**.

1 $0 \leq P(E) \leq 1$

2 $P(S) = 1$, S is the sample space (the sure event)

3 If E_1 and E_2 are mutually exclusive events, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Note:

P is a function with domain the set of subsets of S (Sample space) and its range is the set of real numbers between 0 and 1 (both inclusive). Thus we note the following:

a The probability of an event is always between 0 and 1.

b If $E = \varnothing$ (the impossible event), then $P(\varnothing) = 0$, and if $E = S$ (the certain event), then $P(S) = 1$.

c If $E \cup E' = S$ then $P(E \cup E') = P(S) = 1$, and $P(E') = 1 - P(E)$, where $E' = S \setminus E$ (not E).

Example 33 A box contains 6 red balls. One ball is drawn at random. Find the probability of getting

i a red ball **ii** a white ball

Solution

i The box contains all red balls. Hence we are sure that red will occur. Then, the probability of getting a red ball is one.

That is, $P(R) = \dfrac{n(R)}{n(S)} = \dfrac{6}{6} = 1$

ii The box contains no white balls. The chance of getting white ball is impossible, and the probability is zero.

That is, $P(W) = \dfrac{n(W)}{n(S)} = \dfrac{0}{6} = 0$

Example 34 A bag contains 3 red, 5 black, and 4 white marbles. One marble is drawn at random. What is the probability that the marble is

a black **b** not black

Solution

a $P(\text{black}) = \dfrac{5}{12}$

b $P(\text{not black}) = 1 - P(\text{black})$ . . . . . . complementary events

$= 1 - \dfrac{5}{12} = \dfrac{7}{12}$.

Thus, $P(\text{black}) + P(\text{not black}) = \dfrac{5}{12} + \dfrac{7}{12} = \dfrac{12}{12} = 1$

Example 35 Which of the following cannot be valid assignments of probabilities for outcomes of sample space $S = \{w_1, w_2, w_3\}$ where $w_i \cap w_j = \varnothing$, if $i \neq j$.

	w_1	w_2	w_3
a	0.3	0.6	0.2
b	0.2	0.5	0.3
c	0.3	-0.2	0.9

Solution

a is not valid assignment because the sum of the probabilities is not 1.

b is valid; all the properties in the axiom above are satisfied.

c is not valid because probability cannot be negative.

Odds in favour of and odds against an event

If m and n are probabilities of the occurrence and non-occurrence of an event respectively, then the ratio $m : n$ is called the odds **in favour** of the event.

The ratio $n : m$ is called the **odds against** the event.

Example 36 The odds against a certain event are 5 : 7. Find the probability of its occurrence.

Solution Let E be the event. Then, we are given that number (not E) = 5 and number (E) = 7.

$$n(S) = n(\text{not } E) + n(E) = 5 + 7 = 12$$

$$\therefore\ P(E) = \frac{7}{12}.$$

Example 37 The odds in favour of an event are 3 : 8. Find the probability of its occurrence.

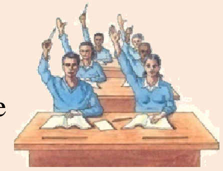
Solution $n(E) = 3$, $n(\text{not } E) = 8$. Thus $n(S) = 3 + 8 = 11$.

$$\therefore\ P(E) = \frac{3}{11}.$$

Rules of probability

In the last section, you have seen different types of events and approaches to probability. We will now discuss some essential rules for probability and probabilities of the different types of events.

ACTIVITY 5.11



For two events E_1 and E_2 discuss what conditions apply to the following rules.

- **a** $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$
- **b** $P(E_1 \cup E_2) = P(E_1) + P(E_2)$
- **c** Illustrate each of the above by using a Venn diagram.

In your previous discussions, you saw how to determine probabilities of events.

Example 38 Find the probability of obtaining a 6 or 4 in one roll of a die.

Solution In one roll of a die, the sample space is S = {1, 2, 3, 4, 5, 6}.

Obtaining 6 or 4 gives the event $E = \{4, 6\}$.

$$\text{Thus } P(4 \text{ or } 6) = P(E) = \frac{\textit{number of outcomes favouring } E}{\textit{number of all possible outcomes}} = \frac{2}{6} = \frac{1}{3}.$$

Trying to calculate probabilities by listing all outcomes and favourable outcomes may not always be convenient. For more complex situations, there are rules we can use to help us calculate probabilities.

Addition rule of probability

From previous discussions, recall that, if $E_1, E_2, \ldots, E_n$ form a set of exhaustive events of a sample space S, then $E_1 \cup E_2 \cup \ldots \cup E_n = S$. Moreover, the probability of an event E, i.e. $P(E)$ is given by

$$P(E) = \frac{\text{number of outcomes favoring E}}{\text{total number of otucomes in the sample space}} = \frac{n(E)}{n(S)}.$$

With these we can easily calculate probabilities of compound events by making use of the addition rule stated below.

Addition rule

If E_1 and E_2 are any two events, then,

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$ and

if the events are mutually exclusive, (i.e., $E_1 \cap E_2 = \varnothing$) then $P(E_1 \cap E_2) = 0$ so that

$P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

Example 39

- **a** Find the probability of obtaining a 6 or 4 in one roll of a die.
- **b** Find the probability of getting Head or Tail in tossing a coin once.
- **c** A die is rolled once. Find the probability that it is even or it is divisible by 3.

Solution

a Let E_1 be event of getting 6, E_2 be event of getting 4.

Then E_1 and E_2 are mutually exclusive events

$$\therefore\ P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$

b The events are mutually exclusive

$$\therefore\ P(H \text{ or } T) = P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1.$$

c $S = \{1, 2, 3, 4, 5, 6\}$

Let E_1 = getting even = $\{2, 4, 6\}$.

E_2 = getting a number divisible by 3 = $\{3, 6\}$.

Then E_1 and E_2 are not mutually exclusive, because $E_1 \cap E_2 = \{6\}$

$\therefore$ P(even or divisible by 3) = P(even) + P(divisible by 3) – P(even and divisible by 3).

$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}.$$

This shows the addition rule of probability with two events. What do you think the rule will be for three or more events? The rule can be extended for a finite number of events, but becomes increasingly complicated. For example, for three events it becomes:

✍Note:

$P(E_1 \cup E_2 \cup E_3)$

$= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$

Multiplication rule of probability

This rule is useful for determining the probability of the joint occurrence of events. It is based on the concepts of independence or dependence of events, discussed earlier. Let us take a brief revision of independent and dependent events.

When the occurrence of the first event affects the occurrence of the second event in such a way that the probability is changed, the events are said to be dependent.

Example 40 A bag contains 3 black and 2 white balls. We draw two balls one after the other with replacement (the second is drawn after the first is replaced). Find the probability that the first ball is black and the second ball is also black.

Solution Let event A be the first ball is black.

Let event B be the second ball is black.

Then $P(A) = \frac{3}{5}$ and $P(B) = \frac{3}{5}$ (*Since the ball is replaced, the sample space is not affected*).

Example 41 Suppose we repeat the experiment in Example 39, but this time the first ball is NOT replaced. This time

$P(A) = P$ (the first ball is black) $= \frac{3}{5}$

If the first ball is black $P(B) = \frac{2}{4}$ (one black ball has been removed)

If the first ball was NOT black $P(B) = \frac{3}{4}$

Recognizing dependence or independence is of paramount importance in using the multiplication rule of probability. When occurrence of one event depends on the occurrence of another event, we say the second event is conditioned by the first event. This leads into what is called **conditional probability**.

Conditional probability

If E_1 and E_2 are two events, the probability that E_2 occurs given that E_1 has already occurred is denoted by $P(E_2 \mid E_1)$ and is called the conditional probability of E_2 given that E_1 has already occurred. If the occurrence or non occurrence of E_1 does not affect the probability of E_2, or if E_1 and E_2 are independent, then $P(E_2 \mid E_1) = P(E_2)$. This is often called the **multiplication rule of probability**.

Multiplication rule of probability

If E_1 and E_2 are any two events, the probability that both events occur, denoted by $P(E_1 \text{ and } E_2) = P(E_1 \cap E_2) = P(E_1 E_2)$ is given by

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2 \mid E_1), \text{ whenever } P(E_1) \neq 0.$$
$$= P(E_2) \times P(E_1 \mid E_2), \text{ whenever } P(E_2) \neq 0.$$

✍Note:

If E_1 and E_2 are independent, then $P(E_2 \mid E_1) = P(E_2)$.

Hence, $P(E_1 \cap E_2) = P(E_1) \times P(E_2)$ for independent events E_1 and E_2.

Example 42

a A box contains 3 red and 2 black balls. One ball is drawn at random, is not replaced, and a second ball is drawn. Find the probability that the first ball is red and the second is black.

b A die is rolled and a coin is tossed. Find the probability of getting 3 on the die and a tail in the coin.

c A bag contains 3 red, 4 blue and 3 white balls. Three balls are drawn one after the other. Find the probability of getting a red ball on the first draw, a blue ball on the second draw and a white ball on the third draw if

i each ball is drawn, but then is replaced back before the next draw.

ii the balls are drawn without replacement.

Solution

a Let E_1 = getting red in the first draw.

E_2 = getting black in the second draw.

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2 \mid E_1) = \frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}.$$

b Let E_1 = getting 3 on the die and E_2 = getting tail on the coin.

Since the two events are independent,

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}.$$

c Let E_1 = getting red, in the first draw,

E_2 = getting blue in the second draw,

E_3 = getting white in the third draw.

i The balls are replaced after each draw. The events are independent.

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \times P(E_2) \times P(E_3) = \frac{3}{10} \times \frac{4}{10} \times \frac{3}{10} = \frac{36}{1000} = \frac{9}{250}.$$

ii The balls are not replaced, so events are dependent .

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \times P(E_2 \mid E_1) \times P(E_3 \mid E_1 \text{ and } E_2) = \frac{3}{10} \times \frac{4}{9} \times \frac{3}{8} = \frac{1}{20}.$$

Exercise 5.15

1 A die is rolled. What is the probability of scoring

a 4 ? **b** 3 or 5?

2 In throwing a die, consider the following events.

E_1 = the number that shows up is even

E_2 = the number that shows up is prime

E_3 = the number that shows up is more than 3

a Determine the event $E_2 \cap E_3$

b Determine the number of elements in $E_1 \cap E_2$

c Determine the number of elements in $P(E_1 \cap E_2 \cap E_3)$

d Determine the $P(E_1 \cap E_2)$

e Determine the $P(E_1 \cup E_2)$

f Determine $P(E_1 \cup E_2 \cup E_3)$

3 From a pack of 52 playing cards, one card is drawn. Find the probability that it is

a either a King or a Jack;

b either a Queen or red.

4 A die is thrown twice. What is the probability of scoring a 3, followed by a 4?

5 A red ball and 4 white balls are in a box. If two balls are drawn without replacement, what is the probability of

a getting a red ball on the first draw and a white ball on the second?

b getting two white balls?

6 Two cards are drawn from a pack of 52 cards. What is the probability that the first is an Ace and the second is a King,

a if the first card was replaced before the second was drawn?

b if the cards were drawn without replacement?

7 A box contains 24 pens, 10 of which are red. A pen is picked at random. What is the probability that the pen is not red?

8 The following table gives assignments of probabilities for outcomes from a sample space.

	w_1	w_2	w_3	w_4	w_5	w_6	w_7
a	0.1	0.001	0.05	0.03	0.01	0.2	0.6
b	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
c	0.1	0.2	0.3	0.4	0.5	0.6	0.7
d	-0.1	0.2	0.3	0.4	-0.2	0.1	0.3
e	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{13}{14}$

a Which of the probabilities are invalid assignments? Why?

b Why is (b) a valid assignment of probabilities.

9 In throwing a die what is the probability of getting even or prime number?

10 Two students are selected from a class of 28 girls and 22 boys one after the other. What is the probability that the second student selected is a boy given that the first was a girl?

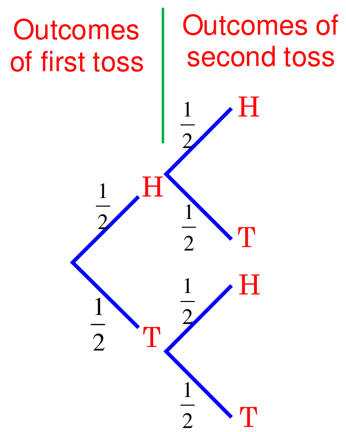
You have seen how to determine probability by using either of the product rules (for independent or dependent events). It is also possible to show joint events using tree diagrams and tables, and calculate probabilities from these.

Example 43 A fair coin is tossed twice. Find the probability that both outcomes will be Heads.

Solution: From the multiplication rule, $P(HH) = P(H) \times P(H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

You can use a tree diagram and/or table to portray the possible outcomes.

Using tree diagram



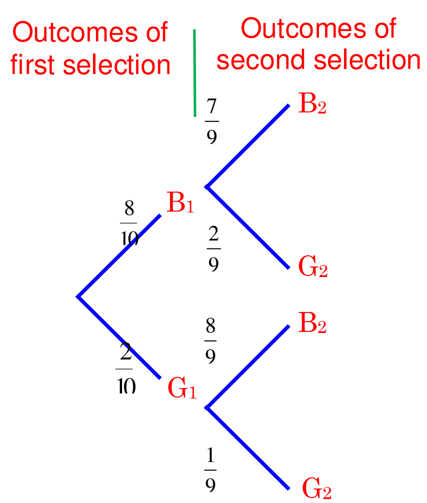
Joint event	Probability of joint event
HH	$\frac{1}{4}$
HT	$\frac{1}{4}$
TH	$\frac{1}{4}$
TT	$\frac{1}{4}$

Therefore, the probability that both outcomes are heads is $\frac{1}{4}$.

Example 44 Suppose that a group of 10 students contain eight boys (B) and two girls (G). If two students are chosen randomly without replacement, find the probability that the two students chosen are both boys.

Solution: $P(B_1 \text{ and } B_2) = P(B_1) \times P(B_2/B_1) = \frac{8}{10} \times \frac{7}{9} = \frac{56}{90} = \frac{28}{45}$.

Hence, the probability that the two students chosen are both boys is $\frac{56}{90}$.



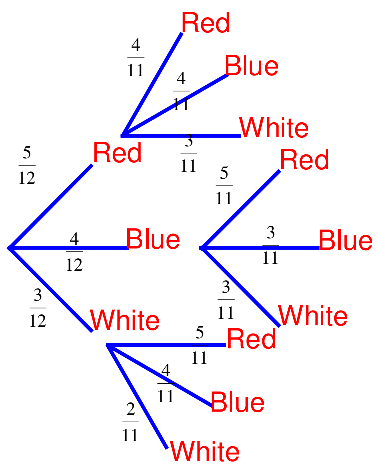
Joint Event	Probability of joint event
B_1 and B_2	$\frac{56}{90}$
B_1 and G_2	$\frac{16}{90}$
G_1 and B_2	$\frac{16}{90}$
G_1 and G_2	$\frac{2}{90}$

Example 45 A bag contains 5 red balls, 4 blue balls, and 3 white balls. Two balls are drawn one after the other, without replacement.

a Find the probability that both are red.

b Draw tree diagram representing the experiment.

Solution: $P(R \text{ and } R) = \frac{5}{12} \times \frac{4}{11} = \frac{20}{132} = \frac{5}{33}$.



Joint Event	Probability of joint Event
R and R	$\frac{5}{12} \times \frac{4}{11}$
R and B	$\frac{5}{12} \times \frac{4}{11}$
R and W	$\frac{5}{12} \times \frac{3}{11}$
B and R	$\frac{4}{12} \times \frac{5}{11}$
B and B	$\frac{4}{12} \times \frac{3}{11}$
B and W	$\frac{4}{12} \times \frac{3}{11}$
W and R	$\frac{3}{12} \times \frac{5}{11}$
W and B	$\frac{3}{12} \times \frac{4}{11}$
W and W	$\frac{3}{12} \times \frac{2}{11}$

Example 46 Two dice are thrown simultaneously. Find the probability that the sum of the numbers scored is

a 7 **b** greater than 9 **c** less than 4

Solution:

		Second die					
		1	2	3	4	5	6
First die	1	(1 , 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

From the table above $n(S) = 36$.

a Let E = the sum of numbers at the top is 7. Then $n(E) = 6$.

$\therefore\ P(E) = \frac{6}{36} = \frac{1}{6}$.

b Let E = sum of the numbers at the top is greater than 9 (i.e., 10 or 11 or 12)

$\therefore\ P(E) = \frac{6}{36} = \frac{1}{6}$.

c Let E = sum is less than 4 (i.e. 2 or 3). Then, $n(E) = 3$

$\therefore\ P(E) = \frac{3}{36} = \frac{1}{12}$.

Exercise 5.16

1 A box contains 5 red and 6 white balls. If one ball is drawn at random, find the probability that it will be

a red or white? **b** not red? **c** yellow?

2 From a pack of 52 playing cards, three cards are drawn one after the other. What is the probability that all are Kings if

a drawing is made with replacement?

b drawing is made without replacement?

3 Use the table in Example 45, to find the probability that

a the sum of the top numbers is 12.

b the sum of the top numbers is 13.

c the sum of the numbers is greater than 10.

4 There are 4 black, 2 red and 4 white balls in a box. If three balls are selected at random what is the probability that

a all the balls selected are black? **b** at least one ball is white?

c all the balls are of different colour?

5 Two lamps are to be chosen from a pack of 12 lamps where four are defective and the rest are non defective. What is the probability that

a both are defective? **b** One is defective?

c at most one is defective?

6 If a plate of a car consists of two letters and four digits and one car is chosen at random, then find the probability that the car has the letters at the beginning and at the end.

Key Terms

class boundary	exhaustive events	percentiles
class interval	frequency	permutation
class limit	fundamental counting principles	probability of an event
class mid point	independent events	qualitative data
combination	mean	quantitative data
continuous variable	measures of location	quartiles
deciles	measures of variations	range
dependent events	median	standard deviation
discrete variable	mode	variance

Summary

1. **Quantitative data** can be numerically described. Height, weight, age, etc. are quantitative.
2. **Qualitative data** cannot be expressed numerically. Honesty, beauty, sex, love, religion, etc. are qualitative.
3. A quantity which assumes different values is said to be a **variable**. A variable may be
   i. **continuous,** if it can take any numerical value within a certain range. Some examples are height, weight, temperature.
   ii. **discrete,** if it takes only discrete or exact values. It is obtained by counting.
4. **Frequency** means the number of times a certain value of a variable is repeated in the given data.
5. A **grouped frequency distribution** is constructed to summarize a large sample of data.

   The appropriate class interval is given by

$$\text{Class interval} = \frac{\begin{pmatrix}\text{Largest value}\\ \text{in ungrouped data}\end{pmatrix} - \begin{pmatrix}\text{Smallest value}\\ \text{in ungrouped data}\end{pmatrix}}{\text{Number of classes required}}$$

6 A **measure of location** is a single value that is used to represent a mass of data. The common measures of location are **mean**, **median**, **mode**, **quartiles**, **deciles** and **percentiles**.

$$\text{Mean } (\bar{x}) = \frac{\sum_{i=1}^{n} x_i}{n} \quad \textit{for raw data}$$

$$= \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} \quad \textit{for discrete data}$$

$$= \frac{\sum_{i=1}^{n} f_i m_i}{\sum_{i=1}^{n} f_i} \quad \textit{for grouped data } (m = \textit{class mark})$$

7 **Median of ungrouped data** is given by

$$M_d = \left(\frac{(n+1)^{th}}{2} item\right), \text{ if } n \text{ is odd}$$

$$= \frac{\left(\frac{n}{2}\right)^{th} item + \left(\frac{n}{2}+1\right)^{th}}{2} item, \quad \text{if } n \text{ is even}$$

(After data is arranged in increasing or decreasing order of magnitude.)

8 **Median for a grouped data** is given by $M_d = B_L + \left(\frac{\frac{n}{2} - cf_b}{f_c}\right) i$

9 **Mode** is the value with the highest frequency.

10 If a distribution has a single mode it is "**unimodal**". If it has two modes, it is "**bimodal**". If it has more than two modes, it is called "**multimodal**".

11 For grouped frequency distributions, the mode is given by $M_o = B_L + \left(\frac{d_1}{d_1 + d_2}\right) i$

12 **Quartiles** for grouped frequency distributions are given by $Q_R = B_L + \left(\frac{\frac{tn}{4} - cf_b}{f}\right) i$

13 Similarly the t^{th} **Decile** and i^{t} Percentile for grouped frequency distributions, are given by

$$D_t = B_L + \left(\frac{\frac{tn}{10} - cf_b}{f}\right) i \text{ and } P_i = B_L + \left(\frac{\frac{tn}{100} - cf_b}{f}\right) i \quad \text{respectivley.}$$

14 **Variation** is used to demonstrate the extent to which the individual item in the distribution varies from the average.

15 The different **measures of variation** are **Range**, **Variance** and **Standard Deviation**.

- ✓ Range $= x_{\max} - x_{\min}$
- ✓ Variance $= \dfrac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n}$
- ✓ Standard deviation (S) is the positive square root of variance,

  $S = \sqrt{Variance}$

16 **Probability of an event E** is defined as follows

If an experiment results in n equally likely outcomes and $m < n$ is the number of the ways favourable for event E, then $P(E) = \dfrac{m}{n}$.

17 ***Multiplication Principle***

If an event can occur in m different ways and for every such choice another event can occur in n different ways, then both events can occur in the given order in $m \times n$ different ways.

18 ***Addition Principle***

If an operation can be performed in m different ways and another operation can occur in n different ways and the two operations are mutually exclusive, (the performance of one excludes the other) then either of the two can be performed in $m + n$ ways.

19 If n is a natural number, then n **factorial**, denoted by $n!$, is defined by

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 \qquad (0! = 1)$$

20 **Permutations** are the number of arrangements of n objects taking r of them at a time is denoted by P(n, r) where $P(n,r)=\frac{n!}{(n-r)!}$.

21 The **number of combination** of n things taking r at a time is given by

$$nCr=\binom{n}{r}=\frac{P(n,\ r)}{r!}=\frac{n!}{(n-r!)\,r!}.$$

22 **The Binomial Theorem:** $(x+y)^n=nC_ox^n+nC_1x^{n-1}y+nC_2x^{n-2}y^2+...+nC_ny^n$.

Review Exercises on Unit 5

1 Construct a grouped frequency distribution table for the following data:

13	1	18	21	2	5	15	17	3	20
15	5	16	12	4	2	1	5	12	10
22	13	18	16	15	9	8	7	6	12
24	16	3	13	17	15	15	4	3	12

Hint:- *Use 8 classes.*

2 Find the mode(s) of each of the following scores

a 10, 4, 3, 6, 4, 2, 3, 4, 5, 6, 8, 10, 2, 1, 4, 3

b 4, 3, 2, 4, 6, 5, 5, 7, 6, 5, 7, 3, 1, 7, 2

c

x	20-39	40-59	60-79	80-99	100-119	120-139	140-159	160-179	180-199
f	6	9	11	14	20	15	10	8	7

3 Find the median of each of the following scores

a 2, 3, 16, 5, 15, 38, 18, 17, 12 **b** 3, 2, 6, 8, 12, 4, 3, 2, 1, 6

c

x	300–309	310–319	320–329	330–339	340–349	350–359	360–369	370–379
f	9	20	24	38	48	27	17	6

4 Find the mean of each of the following scores

a 12, 8, 7, 10, 6, 14, 7, 6, 12, 9 **b** 2.1, 6.3, 7.1, 4.8, 3.2

c

x	12	13	14	15	16	17	18	20
f	4	11	32	21	15	8	5	4

d Find the mean score of 30 students with the following scores in mathematics

Score	Number of students
40 – 49	2
50 – 59	0
60 – 69	6
70 – 79	12
80 – 89	8
90 – 99	2

5 Find Q_2, D_3 and P_{20} of the following.

x	2.5	7.5	12.5	17.5	22.5
f	7	18	25	30	20

6 Find the variance and standard deviation of each of the following scores.

a 3, 5, 7, 8, 2, 11, 6, 5

b

x	3	4	5	6	7
f	2	4	8	4	2

c

x	1 – 3	4 – 6	7 – 9	10 – 12	13 – 15	16 – 18	19 – 21
f	1	9	25	35	17	10	3

7 If a fair coin is tossed 6 times what is the probability that

a 6 heads will occur? **b** 2 heads will occur?

8 If $\frac{(n+1)!}{n!} = 5$, then find n.

9 How many three – digit numbers can be formed from the digits 2, 5, 7, 9

a if each digit is used once only?

b if each may be used repeatedly?

10 Compute

a $_6C_2$ **b** $_8C_6$ **c** $_3C_1$.

11 A box contains 12 bulbs with 3 defective ones. If two bulbs are drawn from the box together, what is the probability that

a both bulbs are defective? **b** both are non defective?

c one bulb is defective?

12 In how many ways can 8 people be arranged at a round table?

13 In the expansion of $(a + b)^{10}$, find

a the coefficient of $a^7 b^3$. **b** the coefficient of $a^3 b^7$.

14 A committee of 5 members is to be selected from 7 men and 8 women. In how many ways can this be done so as to include

a 2 women? **b** at least 2 men? **c** at most 4 women?

15 A box contains 3 red and 8 white balls. If one ball is drawn from it, find the chance that the ball drawn is red.

16 From a pack of 52 playing cards, three cards are drawn one after the other without replacement. What is the probability that Ace, King and Jack will be obtained respectively?

17 Suppose a pair of dice is thrown. What is the probability that the sum of the scores is 5?

Unit 6

MATRICES AND DETERMINANTS

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about matrices.*
- *know specific ideas, methods and principles concerning matrices.*
- *perform operation on matrices.*
- *apply principles of matrices to solve problems.*

Main Contents

INTRODUCTION

Matrices appear wherever information is expressed in tables. One such example is a monthly calendar as shown in the figure, where the columns give the days of the week and the rows give the dates of the month. A matrix is simply a rectangular table or array of numbers written in either ( ) or [ ] brackets. Matrices have many applications in science, engineering and computing. Matrix calculations are used in connection with solving linear equations.

In this unit, you will study matrices, operations on matrices, and determinants. You will also see how you can solve systems of linear equations using matrices.

HISTORICAL NOTE

Arthur Cayley (1821-95)

Many people have contributed to the development of the theory of matrices and determinants. Starting from the 2nd century BC, the Babylonians and the Chinese used the concepts in connection with solving simultaneous equations. The first abstract definition of a matrix was given by Cayley in 1858 in his book named Memoir on the theory of matrices.

He gave a matrix algebra defining addition, multiplication, scalar multiplication and inverses. He also gave an explicit construction of the inverse of a matrix in terms of the determinant of the matrix.

OPENING PROBLEM

Consider a nutritious drink which consists of whole egg, milk and orange juice. The food energy and protein of each of the ingredients are given by the following table.

	Food Energy (Calories)	Protein (Grams)
1 egg	80	6
1 cup of milk	160	9
1 cup of Juice	110	2

How much of each do you need to produce a drink of 540 calories and 25 grams of protein?

6.1 MATRICES

Definition 6.1

Let $\mathbb{R}$ be the set of real numbers and m and n be positive integers.

A rectangular array of numbers in $\mathbb{R}$ of the form,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called an m by n $(m\times n)$ matrix in $\mathbb{R}$.

Consider the matrix A in the definition above:

- ✓ The number m is called the number of rows of A.
- ✓ The number n is called the number of columns of A.
- ✓ The number a_{ij} is called the ij^{th} element or entry of A which is an element in the i^{th} row and j^{th} column of A.
- ✓ A can be abbreviated by: $A = (a_{ij})_{m\times n}$
- ✓ The rectangular array of entries is enclosed in an ordinary bracket or in a square bracket.
- ✓ $m\times n$ (read as m by n) is called the **size or order** of the matrix.

Example 1 Consider the matrix:.

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 4 & 0 & 3 \end{pmatrix}$$

Then A is a 2×3 matrix with $a_{11} = 1$, $a_{13} = 2$ and $a_{23} = 3$.

Example 2 The matrix $A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \\ 4 & 0 \end{pmatrix}$ is a 3×2 matrix with:

$$a_{11} = 3, a_{12} = -1, a_{21} = 1, a_{22} = 2,\ a_{31} = 4 \text{ and } a_{32} = 0.$$

✍Note:

- ✓ The entries in a given matrix need not be distinct.
- ✓ The best way to view matrices is as the contents of a table where the labels of the rows and columns have been removed.

Example 3 Three students Chaltu, Solomon and Kalid have a number of 10, 50 and 25 cent coins in their pockets. The following table shows what they have.

		Student name		
		Chaltu	Kalid	Solomon
No of coins	10 cent coins	2	6	4
	50 cent coins	3	2	0
	25 cent coins	4	0	5

a Represent the table in matrix form.

b What is represented by the columns?

c What is represented by each row?

d Suppose a_{ij} denotes the entry in the i^{th} row and j^{th} column. What does a_{31} tell you? What about a_{23}?

Solution

a $A = \begin{pmatrix} 2 & 6 & 4 \\ 3 & 2 & 0 \\ 4 & 0 & 5 \end{pmatrix}$

b The columns represent the number of the various kinds of coins each student has.

c The rows represent the number of coins of a certain fixed value that the students have.

d $a_{31} = 4$. It means Chaltu has four 25-cent coins in her pocket.

$a_{23} = 0$. This means Solomon has no 50-cent coins.

ACTIVITY 6.1

In each of the following matrices, determine the number of rows and the number of columns.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \\ 29 \end{pmatrix},\ C = \begin{pmatrix} 0 & -5 \\ 3 & 4 \\ 8 & 6 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & -6 & 7 \end{pmatrix}.$$

From Activity 6.1, you may have observed that:

- ✓ The number of rows and columns in matrix A are equal.
- ✓ The number of columns in matrix B is one.
- ✓ The number of rows in matrix D is one.

Some important types of matrices

1. A matrix with only one column is called a **column matrix**. It is also called a **column vector**.
2. A matrix with only one row is called a **row** matrix (also called a row vector).
3. A matrix with the same number of rows and columns is called a **square matrix**.
4. A matrix with all entries 0 is called a **zero** matrix which is denoted by **0**.
5. A **diagonal matrix** is a square matrix that has zeros everywhere except possibly along the main diagonal (top left to bottom right).
6. The **identity** (**unit**) matrix is a diagonal matrix where the elements of the principal diagonal are all ones.
7. A **scalar** matrix is a diagonal matrix where all elements of the principal diagonal are equal.
8. A **lower triangular matrix** is a square matrix whose elements above the main diagonal are all zero.
9. An **upper triangular matrix** is a square matrix whose elements below the main diagonal are all zero.

Example 4 Give the type(s) of each matrix below.

a $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

d $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ **e** $\begin{pmatrix} -35 & 0 & 4 \end{pmatrix}$ **f** $\begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

g $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

a a zero matrix **b** It is a square, zero, diagonal and scalar matrix

c a diagonal matrix **d** a column matrix **e** a row matrix

f a scalar matrix **g** an identity matrix

Example 5 Decide whether each matrix is upper triangular, lower triangular or neither.

a $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 3 & 9 & 7 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}$ **c** $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 4 \\ 0 & 0 & 7 \end{pmatrix}$

d $\begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}$ **e** $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ **f** $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 0 & 0 & 9 \end{pmatrix}$

Solution

a lower triangular **b** lower triangular **c** upper triangular

d upper triangular **e** both (notice that it satisfies both conditions)

f neither

Equality of matrices

Definition6.2

Two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ of the same order are said to be **equal**, written $A = B$, if their corresponding elements are equal, i.e. $a_{ij} = b_{ij}$ for all $1 \le i \le m$ and $1 \le j \le n$.

Example 6 Find x and y if the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & x+y & -1 \\ x & -7 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 1 & -7 & 3+y \end{pmatrix} \text{ are equal.}$$

Solution If $A = B$, then $\begin{cases} x+y=0 \\ x=1 \\ 3+y=2 \end{cases}$

Solving this gives you: $x = 1$ and $y = -1$.

Addition and subtraction of matrices

ACTIVITY 6.2

A school book store has books in four subjects for four grade levels. Some newly ordered books have arrived.

	Previous Books in Stock			
	Grade Level			
	7	8	9	10
Biology	101	89	72	75
Physics	62	58	70	43
Chemistry	57	65	71	94
Mathematics	81	87	91	93

	Newly arrived Books			
	Grade Level			
	7	8	9	10
Biology	60	65	54	45
Physics	27	35	50	27
Chemistry	55	66	65	44
Mathematics	75	68	70	51

How many of each kind do they have now?

Definition 6.3

Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ be two matrices. Then the sum of A and B, denoted by $A + B$, is obtained by adding the corresponding elements, while the difference of A and B, denoted by $A - B$, is obtained by subtracting the corresponding elements i.e., $A + B = (a_{ij} + b_{ij})_{m\times n}$ and $A - B = (a_{ij} - b_{ij})_{m\times n}$.

Example 7 Let $A = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 4 & 1 \\ 6 & 0 & 3 \\ 3 & 6 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & 4 \\ 5 & 0 & 3 \\ 6 & 0 & 2 \\ 4 & 0 & 4 \end{pmatrix}$.

Find the sum and difference of A and B, if they exist.

Solution $A + B = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 4 & 1 \\ 6 & 0 & 3 \\ 3 & 6 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 4 \\ 5 & 0 & 3 \\ 6 & 0 & 2 \\ 4 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 5+3 & 2+1 & 2+4 \\ 4+5 & 4+0 & 1+3 \\ 6+6 & 0+0 & 3+2 \\ 3+4 & 6+0 & 0+4 \end{pmatrix} = \begin{pmatrix} 8 & 3 & 6 \\ 9 & 4 & 4 \\ 12 & 0 & 5 \\ 7 & 6 & 4 \end{pmatrix}$

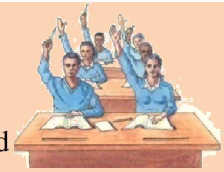
$$A - B = \begin{pmatrix} 5 & 2 & 2 \\ 4 & 4 & 1 \\ 6 & 0 & 3 \\ 3 & 6 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 1 & 4 \\ 5 & 0 & 3 \\ 6 & 0 & 2 \\ 4 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -2 \\ -1 & 4 & -2 \\ 0 & 0 & 1 \\ -1 & 6 & -4 \end{pmatrix}$$

Example 8 Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 7 & 9 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$.

Find $A - B$ and $B + C$, if they exist.

Solution $A - B = \begin{pmatrix} -1 & 1 & 0 \\ 6 & -2 & -5 \end{pmatrix}$, but since B and C have different orders, they cannot be added together.

ACTIVITY 6.3



Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 7 & -3 \\ 2 & 5 \end{pmatrix}$ and $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Find

a $(A + B) + C$, **b** $A + (B + C)$ **c** $A - A$

d $A + 0$ **e** $A + B$ **f** $B + A$

From Activity 6.3, you can observe the following properties of matrix addition.

1. $A + B = B + A$ (Commutative property)
2. $(A + B) + C = A + (B + C)$ (Associative property)
3. $A + 0 = A = 0 + A$ (Existence of additive identity)
4. $A + (-A) = 0$ (Existence of additive inverse)

Multiplication of a matrix by a scalar

ACTIVITY 6.4

The marks obtained by Nigist and Hagos (out of 50) in their examinations are given below.

	Nigist	Hagos
English	37	31
Mathematics	46	44
Biology	28	25

If the marks are to be converted out of 100, then find the marks of Nigist and Hagos in each subject out of 100.

From Activity 6.4, you may have observed that given a matrix, you can get another matrix by multiplying each of its elements by a constant.

Definition 6.4

If r is a scalar (i.e. a real number) and $A = (a_{ij})_{m\times n}$ is a given matrix, then rA is the matrix obtained from A by multiplying each element of A by r. i.e $rA = (ra_{ij})_{m\times n}$

Example 9 If $A = \begin{pmatrix} 5 & -2 & -2 \\ 4 & 4 & -6.5 \end{pmatrix}$, then find $5A$, $\frac{1}{2}A$ and $-3A$.

Solution $5A=\begin{pmatrix}5\times5 & 5\times(-2) & 5\times(-2)\\ 5\times4 & 5\times4 & 5\times(-6.5)\end{pmatrix}=\begin{pmatrix}25 & -10 & -10\\ 20 & 20 & -32.5\end{pmatrix}$

$$\frac{1}{2}A=\begin{pmatrix}\frac{1}{2}\times5 & \frac{1}{2}\times(-2) & \frac{1}{2}\times(-2)\\ \frac{1}{2}\times4 & \frac{1}{2}\times4 & \frac{1}{2}\times(-6.5)\end{pmatrix}=\begin{pmatrix}\frac{5}{2} & -1 & -1\\ 2 & 2 & -3.25\end{pmatrix} \text{ and}$$

$$-3A=\begin{pmatrix}(-3)\times5 & (-3)\times(-2) & (-3)\times(-2)\\ (-3)\times4 & (-3)\times4 & (-3)\times(-6.5)\end{pmatrix}=\begin{pmatrix}-15 & 6 & 6\\ -12 & -12 & 19.5\end{pmatrix}$$

Example 10 Alemitu purchased coffee, sugar, wheat flour, and teff flour from a shop as shown by the following matrix. Assume the quantities are in kg.

$A=\begin{pmatrix}6\\11\\60\\90\end{pmatrix}$. Find the new matrix, if

a she doubles her order **b** she halves her order

c she orders 75% of her previous order

Solution

a $2A=\begin{pmatrix}12\\22\\120\\180\end{pmatrix}$ **b** $\frac{1}{2}A=\begin{pmatrix}3\\5.5\\30\\45\end{pmatrix}$ **c** $0.75A=\begin{pmatrix}4.5\\8.25\\45\\67.5\end{pmatrix}$

ACTIVITY 6.5

Let $A=\begin{pmatrix}-1 & 1 & -1\\ 6 & -2 & -1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 0 & 1\\ 2 & -1 & 3\end{pmatrix}$

If $r=-7$ and $s=4$, then find each of the following:

a $r(A+B)$ **b** $rA+rB$ **c** $(rs)A$ **d** $r(sA)$

e $(r+s)A$ **f** $rA+sA$ **g** $1A$ **h** $0A$

Properties of scalar multiplication

If A and B are matrices of the same order and r and s are any scalars (i.e., real numbers), then:

a $r(A+B)=rA+rB$ **b** $(r+s)A=rA+sA$

c $(rs)A=r(sA)$ **d** $1A=A$ *and* $0A=0$

Exercise 6.1

1 If $A = \begin{pmatrix} 8 & 2 & 4.23 & -4 \\ 9 & 2 & 1 & 3 \\ 7.5 & 51 & 2 & 4 \\ 0 & 9 & 3 & 6 \end{pmatrix}$, then determine the values of the following:

a a_{21} **b** a_{33} **c** a_{42} **d** a_{32}

2 What is the order of each of the following matrices?

a $\begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 4 & 7 \\ 5 & -6 & 3 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 3 \end{pmatrix}$

d $(1 \;\; 2 \;\; 3)$ **e** (7)

3 What are the diagonal elements of each of the following square matrices?

a $\begin{pmatrix} 1 & 0 & 0 \\ 3 & -4 & 7 \\ 0 & 7 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 1 & 3 & 1 \\ -4.5 & 1 & 8 & 2 \\ 54 & 1 & 71 & 3 \\ 2 & 1 & 5 & 4 \end{pmatrix}$

4 Construct a 3×4 matrix $A = (a_{ij})$, where $a_{ij} = 3i - 2j$.

5 Given $A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$, find each of the following.

a $A + B$ **b** $A - B$ **c** $3B + 2A$

d $B + A$ **e** $2A + 3B$

6 Given $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$, find matrices C that satisfy the following condition:

a $A + C = B$ **b** $A + 2C = 3B$

7 Graduating students from a certain high school sold cinema tickets on two different occasions, in two kebeles, in order to raise money that they wanted to donate to their school. The following matrices show the number of students who attended the occasions.

$$\begin{array}{ccc} & 1^{st} \text{ occasion} & \\ & kebele\ 1 \quad kebele\ 2 & \\ \begin{matrix} Boys \\ Girls \end{matrix} & \begin{pmatrix} 175 & 221 \\ 199 & 150 \end{pmatrix} \end{array} \qquad \begin{array}{ccc} & 2^{nd} \text{ occasion} & \\ & kebele\ 1 \quad kebele\ 2 & \\ \begin{matrix} Boys \\ Girls \end{matrix} & \begin{pmatrix} 120 & 150 \\ 199 & 181 \end{pmatrix} \end{array}$$

a Give the sum of the matrices.

b If the tickets were sold for Birr 2.50 a piece on the 1st occasion and Birr 3.00 a piece on the second occasion, how much money was raised from the boys? from the girls? In kebele 1. What is the total amount raised for the school?

Multiplication of matrices

To study the rule for multiplication of matrices, let us define the rule for matrices of order $1 \times p$ and $p \times 1$.

Let $A = (a_{11} \ \ a_{12} \ \dots \ a_{1p})$ and $B = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{pmatrix}$.

Then the product AB in the given order is the 1×1 matrix given by

$$AB = (a_{11} \ \ a_{12} \ \dots \ a_{1p}) \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{p1} \end{pmatrix} = (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + \dots + a_{1p}b_{p1})$$

Example 11 If $A = (1 \ \ 2 \ \ 3)$ and $B = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$, find AB.

Solution $AB = (1 \ \ 2 \ \ 3) \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = (1 \times 2) + (2 \times (-3)) + (3 \times 1) = -1.$

Note:

✓ The number of columns of A = The number of rows of $B = p$.

✓ The operation is done row by column in such a way that each element of the row is multiplied by the corresponding element of the column and then the products are added.

Notation:

Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m_2} & \dots & a_{mn} \end{pmatrix}$.

Then you denote the i^{th} row and the j^{th} column of A by A_i and A^j, respectively.

Example 12 Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ -3 & 5 & 6 \end{pmatrix}$. Then $A_1 = (1\ 2\ 3)$, $A_2 = (0\ \ 4\ \ 1)$,

$A_3 = (-3\ \ 5\ \ 6)$, $A^1 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$, $A^2 = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$.

ACTIVITY 6.6

Given $A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 3 & 3 \\ 2 & 4 & 2 \\ 2 & 1 & 2 \end{pmatrix}$, find:

a A_1B^1 **b** A_1B^2 **c** A_1B^3

d A_2B^1 **e** A_2B^2 **f** A_2B^3

The matrix $\begin{pmatrix} A_1B^1 & A_1B^2 & A_1B^3 \\ A_2B^1 & A_2B^2 & A_2B^3 \end{pmatrix}$ in Activity 6.6 is the product of A and B, denoted by AB.

In general, you have the following definition of multiplication of matrices.

Definition 6.5

Let $A = (a_{ij})$ be an $m \times p$ matrix and $B = (b_{jk})$ be a $p \times n$ matrix such that the number of columns of A is equal to the number of rows of B. Then the product AB is a matrix $C = (c_{ik})$ of order $m \times n$, where $c_{ik} = A_iB^k$, i.e. $c_{ik} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + a_{i3}b_{3k} + \dots + a_{ip}b_{pk}$

Example 13 Let $A = \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 & -4 \\ 3 & 2 & 6 \end{pmatrix}$. Then find AB

Solution $AB = \begin{pmatrix} A_1B^1 & A_1B^2 & A_1B^3 \\ A_2B^1 & A_2B^2 & A_2B^3 \end{pmatrix}$

$$AB = \begin{pmatrix} (2 \quad 3)\begin{pmatrix}2\\3\end{pmatrix} & (2 \quad 3)\begin{pmatrix}5\\2\end{pmatrix} & (2 \quad 3)\begin{pmatrix}-4\\6\end{pmatrix} \\ (2 \quad -1)\begin{pmatrix}2\\3\end{pmatrix} & (2 \quad -1)\begin{pmatrix}5\\2\end{pmatrix} & (2 \quad -1)\begin{pmatrix}-4\\6\end{pmatrix} \end{pmatrix} = \begin{pmatrix} 13 & 16 & 10 \\ 1 & 8 & -14 \end{pmatrix}$$

ACTIVITY 6.7

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ 4 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & -4 \\ 0 & 1 \end{pmatrix}$. Find:

a $A(BC)$ **b** $(AB)C$ **c** $A(B + C)$

d $AB + AC$ **e** $(B + C)A$ **f** $BA + CA$

Properties of Multiplication of Matrices

If A, B and C have the right order for multiplication and addition i.e., the operations are defined for the given matrices, the following properties hold:

1 $A\ (BC) = (AB)\ C$ (Associative property)

2 $A\ (B + C) = AB + AC$ (Distributive property)

3 $(B + C)\ A = BA + CA$ (Distributive property)

Example 14 Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Find AB and BA.

Solution: $AB = \begin{pmatrix} 3 & 3 \\ 7 & 7 \end{pmatrix}$ and $BA = \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix}$.

From Example 14, you can conclude that multiplication of matrices is not commutative.

Transpose of a matrix

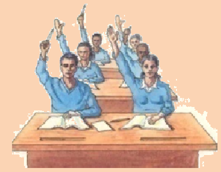
Definition 6.6

The **Transpose** of a matrix $A = (a_{ij})_{m \times n}$, denoted by A^T, is the $n \times m$ matrix found by interchanging the rows and columns of A. i.e., $A^T = B = (b_{ji})$ of order $n \times m$ such that $b_{ji} = a_{ij}$.

Example 15 Give the transpose of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

Solution $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

ACTIVITY 6.8



Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 1 & 0 \\ 1 & 3 \\ 2 & 0 \end{pmatrix}$, find:

a A^{T} **b** $(A^{\mathrm{T}})^{\mathrm{T}}$ **c** $3A^{\mathrm{T}}$

d $(3A)^{\mathrm{T}}$ **e** $(AB)^{\mathrm{T}}$ **f** $B^{\mathrm{T}}A^{\mathrm{T}}$

Properties of transposes of matrices

The following are properties of transposes of matrices:

a $(A^T)^T = A$

b $(A + B)^T = A^T + B^T$, A and B being of the same order.

c $(rA)^T = rA^T$, r any scalar

d $(AB)^T = B^T A^T$; provided AB is defined

Definition 6.7

A square matrix A is called a **symmetric** matrix if $A^T = A$.

Example 16 Show that $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix}$ is symmetric.

Solution $A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix} = A$. So, A is symmetric.

Example 17 Which of the following are symmetric matrices?

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & 4 \\ -2 & 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} a & d & c & d \\ d & k & l & m \\ c & l & w & a \\ d & m & a & x \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 7 & 0 \\ -3 & -1 & 0 \\ 1 & 0 & 5 \end{pmatrix}$$

Solution A and B are symmetric while C is not.

Exercise 6.2

1 Find the products, AB and BA, whenever they exist.

a $A=\begin{pmatrix}3 & 1\\ 3 & -1\end{pmatrix}$, $B=\begin{pmatrix}2 & 1 & -3\\ 3 & 1 & 6\end{pmatrix}$ b $A=\begin{pmatrix}2 & 2\end{pmatrix}$, $B=\begin{pmatrix}1 & 5\\ -2 & 3\\ 0 & 4\end{pmatrix}$

c $A=\begin{pmatrix}-1 & 2\\ 1 & 4\\ -3 & 0\end{pmatrix}$, $B=\begin{pmatrix}1 & -2 & -5\\ 3 & 4 & 0\end{pmatrix}$ d $A=\begin{pmatrix}10 & 3 & 2\\ -8 & -5 & 9\\ -5 & 7 & 7\end{pmatrix}$, $B=\begin{pmatrix}3\\ -1\\ 1\end{pmatrix}$

2 Let $A=\begin{pmatrix}2 & -1 & 3\\ 1 & -1 & 0\\ 0 & 0 & 3\end{pmatrix}$ and $B=\begin{pmatrix}1 & -4\\ 2 & 3\\ 4 & 0\end{pmatrix}$

a What is the order of AB?   b If $C = AB$, then find C_{32}, C_{11} and C_{21}.

3 For the matrices in question 2 above, find $-4AB$, AA, and $A(AB)$.

4 The first of the following tables gives the point system used in soccer (football) in the old days and the point system that is in use now. The second table gives the overall results of 4 teams in a game season.

	Points	
	Old system	New system
Win	2	3
Draw	1	1
Loss	0	0

		Win	Draw	Loss
Teams	A	5	2	2
	B	3	6	0
	C	4	4	1
	D	6	0	3

Let $T=\begin{pmatrix}5 & 2 & 2\\ 3 & 6 & 0\\ 4 & 4 & 1\\ 6 & 0 & 3\end{pmatrix}$ and $P=\begin{pmatrix}2 & 3\\ 1 & 1\\ 0 & 0\end{pmatrix}$. Answer the following questions:

a Find the product TP. Which system is better to rank the teams-the old or the new?

b Which team stands first? Which stands last?

5 If $A = \begin{pmatrix} 3 & -1 \\ 0 & \frac{4}{3} \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 4 & -2 \\ -1 & 0 & 1 \end{pmatrix}$, then find $A + A^T$ and $B + B^T$. Check whether or not the resulting matrices are symmetric.

6 If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then show that $AA^T = A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

7 Show that, if A is a square matrix of order n, then $A + A^T$ is a symmetric matrix. (Hint: Show that $(A + A^T)^T = A^T + A$)

8 A square matrix A is called skew-symmetric, if and only if $A + A^T = 0$. Verify that the following matrices are skew-symmetric:

a $A = \begin{pmatrix} 0 & -1 & 4 \\ 1 & 0 & 7 \\ -4 & -7 & 0 \end{pmatrix}$ **b** $B = \begin{pmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{pmatrix}$

9 If A is a square matrix, show that $A - A^T$ is a skew-symmetric matrix.

10 If A is a skew-symmetric matrix, show that the elements in the main diagonal are all zero.

6.2 DETERMINANTS AND THEIR PROPERTIES

The determinant of a square matrix is a real number associated with the square matrix. It is helpful in solving simultaneous equations. The determinant of a matrix A is associated with A according to the following definition.

Determinants of 2 × 2 matrices

Definition 6.8

1 The determinant of a 1×1 matrix $A = (a)$ is the real number a.

2 The determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined to be the number $ad - bc$.

The determinant of A is denoted by det(A) or $|A|$.

Thus, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Example 1 Find $|A|$ for $A = \begin{pmatrix} 1 & 2 \\ 6 & 4 \end{pmatrix}$.

Solution $|A| = \begin{vmatrix} 1 & 2 \\ 6 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 6 = 4 - 12 = -8$

Note:

- |A| denotes determinant when A is a matrix; the same symbol is used for absolute value of a real number. It is the context that decides the meaning.
- $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denotes a matrix, while $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ denotes its determinant. The determinant is a real number.

ACTIVITY 6.9

Let $A = \begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 \\ 3 & 2 \end{pmatrix}$.

1 Calculate

a $|A|$ **b** $|B|$ **c** $|A^T|$

2 Calculate and compare $|AB|$ and $|A|\,|B|$.

3 Calculate and compare $|A + B|$ and $|A| + |B|$.

Determinants of 3 × 3 matrices

To define the determinant of a 3 × 3 matrix, it is useful to first define the concepts of minor and cofactor.

Let $A = (a_{ij})_{3\times3}$. Then the matrix A_{ij} is a 2×2 matrix which is found by crossing out the i^{th} row and the j^{th} column of A.

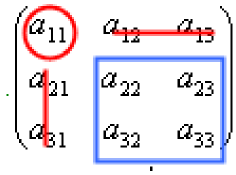
Example 2 If $A = \begin{pmatrix} 0 & 1 & 2 \\ -2 & 3 & 5 \\ 4 & 7 & 18 \end{pmatrix}$, then $A_{11} = \begin{pmatrix} 3 & 5 \\ 7 & 18 \end{pmatrix}$ and $A_{23} = \begin{pmatrix} 0 & 1 \\ 4 & 7 \end{pmatrix}$.

Definition 6.9

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Then $M_{ij} = |A_{ij}|$ is called the **minor** of the element a_{ij} and $c_{ij} = (-1)^{i+j}|A_{ij}|$ is called the **cofactor** of the element a_{ij}.

Example 3 Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Give the minors and cofactors of a_{11}, a_{23} and a_{32}.

Solution The minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$. It is found by crossing out the first row and the first column as in the figure.



Thus, the minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$

The cofactor of $a_{11} = c_{11} = (-1)^{1+1}M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The minor of $a_{23} = M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$, while $c_{23} = (-1)^{2+3}M_{23} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$.

$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ and $c_{32} = -M_{32} = -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$.

Example 4 Find the minors and cofactors of the entries a_{22}, a_{33} and a_{12} of the matrix $\begin{pmatrix} -3 & 4 & -7 \\ 1 & 2 & 0 \\ -4 & 8 & 11 \end{pmatrix}$.

Solution

$$M_{22} = \begin{vmatrix} -3 & -7 \\ -4 & 11 \end{vmatrix} = -61 \text{ and } c_{22} = (-1)^{2+2}M_{22} = \begin{vmatrix} -3 & -7 \\ -4 & 11 \end{vmatrix} = (-3)(11) - (-4)(-7) = -61$$

$$M_{33} = \begin{vmatrix} -3 & 4 \\ 1 & 2 \end{vmatrix} = -10 \text{ and } c_{33} = (-1)^{3+3}M_{33} = \begin{vmatrix} -3 & 4 \\ 1 & 2 \end{vmatrix} = (-3)(2) - (1)(4) = -10$$

$$M_{12} = \begin{vmatrix} 1 & 0 \\ -4 & 11 \end{vmatrix} = 11 \text{ and } c_{12} = (-1)^{1+2}M_{12} = -\begin{vmatrix} 1 & 0 \\ -4 & 11 \end{vmatrix} = -11$$

✍Note:

Note that the 'sign' $(-1)^{i+j}$ accompanying the minors form a chess board pattern with '+' s on the main diagonal as shown : $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

You can now define the 3×3 determinant (determinant of order 3) as follows:

Definition 6.10

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Then the determinant of A along any row i or any column j is given by one of the formulas:

i^{th} row expansion: $|A| = a_{i1}c_{i1} + a_{i2}c_{i2} + a_{i3}c_{i3}$, for any row i (i = 1,2 or 3), or

j^{th} column expansion: $|A| = a_{1j}c_{1j} + a_{2j}c_{2j} + a_{3j}c_{3j}$, for any column j (j = 1,2 or 3).

Note:

Note that the definition states that to find the determinant of a square matrix:

- ✓ choose a row or column;
- ✓ multiply each entry in it by its cofactor;
- ✓ add up these products.

Example 5 Find the determinant of the following matrix A first by expanding along the 1st row and then expanding along the 2nd column, where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ -3 & 2 & 5 \end{pmatrix}$$

Solution

Along row 1:

$$|A| = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = 2(-1)^2 \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 1 & 4 \\ -3 & 5 \end{vmatrix} + 0(-1)^4 \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix}$$

$$= 2(1\times5 - 2\times4) + (-1)(1\times5 - 4\times(-3)) + 0(1\times2 - 1\times(-3))$$

$$= 2(-3) - 1(17) + 0(5) = -6 - 17 = -23$$

$\therefore |A| = -23$

Along Column 2:

$$|A| = a_{12}c_{12} + a_{22}c_{22} + a_{32}c_{32} = 1(-1) \begin{vmatrix} 1 & 4 \\ -3 & 5 \end{vmatrix} + 1(1) \begin{vmatrix} 2 & 0 \\ -3 & 5 \end{vmatrix} + 2(-1) \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$$

$$= -1(1\times5 - 4\times(-3)) + 1(2\times5 - 0\times(-3)) - 2(2\times4 - 0\times1)$$

$$= -1(17) + 1(10) - 2(8) = -17 + 10 - 16 = -23$$

$\therefore |A| = -23,$

Both methods give the same result.

Group Work 6.1

For the matrix $A = \begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{pmatrix}$ do each of the following in groups:

1 a Calculate $|A|$ and $|A^T|$

b What can you conclude from these results?

2 Let B be the matrix found by interchanging row 1 and row 3 of matrix A, i.e.,

$$B = \begin{pmatrix} 2 & 5 & 2 \\ 4 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

a Find $|B|$

b Compare it with $|A|$. What relationship do you see between $|B|$ and $|A|$?

3 Let C be the matrix found by multiplying row 2 by 5. i.e.,

$$C = \begin{pmatrix} 1 & 3 & 2 \\ 5\times4 & 5\times1 & 5\times3 \\ 2 & 5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 20 & 5 & 15 \\ 2 & 5 & 2 \end{pmatrix}$$

a Find $|C|$

b Compare it with $|A|$. What relationship do you see between $|C|$ and $|A|$?

4 Let D be the matrix found by adding 10 times column 1 on column 3. i.e.,

$$D = \begin{pmatrix} 1 & 3 & 2+10\times1 \\ 4 & 1 & 3+10\times4 \\ 2 & 5 & 2+10\times2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 12 \\ 4 & 1 & 43 \\ 2 & 5 & 22 \end{pmatrix}$$

a Find $|D|$

b Compare it with $|A|$. What relationship do you see between $|D|$ and $|A|$?

Properties of determinants

The following properties hold. All the matrices considered are square matrices:

1 $|A| = |A^T|$

Verify this property by considering a 2×2 matrix.

i.e., if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Hence, $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. Also, $|A^T| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

Therefore, $|A| = |A^T|$.

2 If B is found by interchanging two rows (columns) of A, then $|B| = -|A|$.

3 If B is found by multiplying one row (one column) of A by a scalar r, then $|B| = r|A|$.

4 If B is a matrix obtained by adding a multiple of a row (column) of A to another row (column) of A, then $|B| = |A|$.

5 If A has a row (or a column) of zeros, then the determinant of A is zero.

6 If A has two identical rows (or columns), then the determinant of A is zero.

We omit the proofs of the above properties; however, we shall illustrate these properties with examples.

Example 6 Compute the determinant of $\begin{pmatrix} 4 & 0 & -5 \\ 10 & 0 & 7 \\ -14 & 0 & 1 \end{pmatrix}$

Solution By expanding using the 2nd column, we get

$$\begin{vmatrix} 4 & 0 & -5 \\ 10 & 0 & 7 \\ -14 & 0 & 1 \end{vmatrix} = -0\begin{vmatrix} 10 & 7 \\ -14 & 1 \end{vmatrix} + 0\begin{vmatrix} 4 & -5 \\ -14 & 1 \end{vmatrix} - 0\begin{vmatrix} 4 & -5 \\ 10 & 7 \end{vmatrix} = 0$$

Example 7 If $\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix} = 2$, give the values of each of the following.

a $\begin{vmatrix} p & x & p \\ q & y & q \\ r & z & r \end{vmatrix}$ **b** $\begin{vmatrix} p & x & a \\ q & y & b \\ r & z & c \end{vmatrix}$ **c** $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$

d $\begin{vmatrix} p & x & 0 \\ q & y & 0 \\ r & z & 0 \end{vmatrix}$ **e** $\begin{vmatrix} 4a & 12x & 4p \\ b & 3y & q \\ c & 3z & r \end{vmatrix}$ **f** $\begin{vmatrix} a & x & p \\ b & y & q \\ 3b+c & 3y+z & 3q+r \end{vmatrix}$

Solution:

a 0 (1st column and 3rd column are the same.)

b –2 (Column interchange results in change of sign.)

c 2 (A matrix and its transpose have the same determinant.)

d 0 (0 column.)

e 24 (factor 4 out and then 3; $12 \times$ original determinant.)

f 2 (Adding a constant multiple of a row on another row gives the same result.)

Exercise 6.3

1 Compute each of the following determinants:

a $\begin{vmatrix} 1 & 5 \\ 7 & 3 \end{vmatrix}$ **b** $\begin{vmatrix} 1 & 3 & 3 \\ 0 & 2 & -1 \\ 2 & 1 & 2 \end{vmatrix}$ **c** $\begin{vmatrix} a-b & a \\ a & a+b \end{vmatrix}$

2 Solve each of the following equations:

a $\begin{vmatrix} 2x & x \\ 4 & x \end{vmatrix} = 0$ **b** $\begin{vmatrix} 2 & -2 & 1 \\ x & 1 & 0 \\ 3 & 1 & 2 \end{vmatrix} = 1$ **c** $\begin{vmatrix} x+1 & 2 & 1 \\ 1 & 1 & 2 \\ x-1 & 1 & x \end{vmatrix} = 0$

3 For the given matrix A, calculate the cofactor of the given entry:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 9 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix}$$

a a_{32} **b** a_{22} **c** a_{23}

4 **a** Compute the determinant $\begin{vmatrix} 1 & x & y \\ 1 & a & b \\ 1 & c & d \end{vmatrix}$

b Verify that the equation of a straight line through the distinct points (a, b) and (c, d) is given by $\begin{vmatrix} 1 & x & y \\ 1 & a & b \\ 1 & c & d \end{vmatrix} = 0$

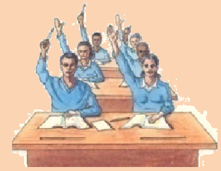
5 Verify that each of the following statements is true. (*Assume that all letters represent non-zero real number*).

a $\begin{vmatrix} x & t+w \\ y & s+u \end{vmatrix} = \begin{vmatrix} x & t \\ y & s \end{vmatrix} + \begin{vmatrix} x & w \\ y & u \end{vmatrix}$ **b** $\begin{vmatrix} a+rb & b \\ c+rd & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

c $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$

6.3 INVERSE OF A SQUARE MATRIX

ACTIVITY 6.10



Let $A=\begin{pmatrix}-3 & 2\\ 1 & -1\end{pmatrix}$, $B=\begin{pmatrix}6 & 2\\ 3 & 1\end{pmatrix}$ and $I_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Find:

a AI_2 **b** I_2A

c Find a matrix C(if it exists) such that $AC = I_2$

d Is there a matrix D so that $BD = I_2$?

From Activity 6.10, the matrix C obtained in (c) is called the inverse of matrix A.

Definition 6.11

A square matrix A is said to be **invertible** or **non-singular**, if and only if there is a square matrix B such that $AB = BA = I$, where I is the identity matrix that has the same order as A.

Remark

The inverse of a square matrix, if it exists, is unique.

Proof: Let A be an invertible square matrix. Suppose B and C are inverses of A.

Then $AB = BA = I$. and $AC = CA = I$ (by definition of inverse)

Now, $B = BI = B(AC) = (BA)C = IC = C$.

Hence, the inverse of A is unique.

Note:

- ✓ Only a square matrix can have an inverse.
- ✓ The inverse of matrix A, whenever it exists, is denoted by A^{-1}.
- ✓ A and A^{-1} have the same order.
- ✓ A matrix that does not have an inverse is called **singular**.

Example 1

a Show that $\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}$ and $\begin{pmatrix}2 & -1\\ -5 & 3\end{pmatrix}$ are inverses of each other.

b Given $A=\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}$, find A^{-1} (if it exists.)

Solution

a $\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\ -5 & 3\end{pmatrix}=\begin{pmatrix}2 & -1\\ -5 & 3\end{pmatrix}\begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Thus, they are inverses of each other.

b Suppose $A^{-1}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $AA^{-1}=I_2$.

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a+c & b+d \\ 2a+3c & 2b+3d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\Rightarrow \begin{cases} a+c=1 \\ 2a+3c=0 \end{cases} \text{ and } \begin{cases} b+d=0 \\ 2b+3d=1 \end{cases}$$

Solving these gives you, $a=3, b=-1, c=-2$ and $d=1$.

Hence $A^{-1}=\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

In the above example, you have seen how to find the inverses of invertible matrices. Sometimes, this method is tiresome and time consuming. There is another method of finding inverses of invertible matrices, using the adjoint.

Definition 6.12

The **adjoint** of a square matrix $A = (a_{ij})$ is defined as the transpose of the matrix $C = (c_{ij})$ where c_{ij} are the cofactors of the elements a_{ij}. Adjoint of A is denoted by **adj** A, i.e., adj $A = (c_{ij})^{\mathrm{T}}$.

Example 2 Find adj A if $A=\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 4 & 0 & 0 \end{pmatrix}$.

Solution

$c_{11}=(-1)^{1+1}\begin{vmatrix} 3 & -1 \\ 0 & 0 \end{vmatrix}=0$, $\quad c_{12}=(-1)^{2+1}\begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix}=-4$,

$c_{13}=(-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix}=-12$, $\quad c_{21}=(-1)^{2+1}\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}=0$,

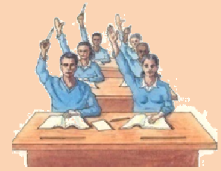
$c_{22}=(-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix}=-4$, $\quad c_{23}=(-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 4 & 0 \end{vmatrix}=0$,

$c_{31}=(-1)^{3+1}\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix}=-3$, $\quad c_{32}=(-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}=3$,

$c_{33}=(-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}=3$.

Then matrix $C=\begin{pmatrix} 0 & -4 & -12 \\ 0 & -4 & 0 \\ -3 & 3 & 3 \end{pmatrix}^{\mathrm{T}}$ and, adj $A=C^T=\begin{pmatrix} 0 & 0 & -3 \\ -4 & -4 & 3 \\ -12 & 0 & 3 \end{pmatrix}$

ACTIVITY 6.11



1 Show that $\text{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

2 Show that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \text{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3 If $A = \begin{pmatrix} 5 & -3 \\ 4 & 2 \end{pmatrix}$, then

a find A^{-1}. **b** find adj$A$.

c find $|A|$. **d** compare A^{-1} and $\frac{1}{|A|}\text{adj}\,A$.

From Activity 6.11, you may have observed that for a 2×2 matrix A,

$$A(\text{adj}A) = |A|\,I_2 = (\text{adj}A)A.$$

If $|A| \neq 0$, then $A\frac{1}{|A|}\text{adj}\,A = I_2$

Therefore, $A^{-1} = \frac{1}{|A|}\text{adj}\,A$

Theorem 6.1

A square matrix A is invertible, if and only if $|A| \neq 0$. If A is invertible, then $A^{-1} = \frac{1}{|A|}\text{adj}\,A.$

Example 3 Find the inverse of $A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 1 \\ -4 & 5 & 2 \end{pmatrix}$

Solution First find adjA.

$c_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = -1$; $c_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ -4 & 2 \end{vmatrix} = -4$; $c_{13} = +\begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix} = 8$

$c_{21} = -\begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} = 19$; $c_{22} = +\begin{vmatrix} 1 & 3 \\ -4 & 2 \end{vmatrix} = 14$; $c_{23} = -\begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = 3$

$c_{31} = +\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} = -8$; $c_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = -1$; $c_{33} = +\begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = 2$

Thus, adj $A = \begin{pmatrix} -1 & 19 & -8 \\ -4 & 14 & -1 \\ 8 & 3 & 2 \end{pmatrix}$

Next, find $|A|$.

$|A| = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = (-1)(-1) + (-2)(-4) + (3)(8) = 31$. Since

$|A| \neq 0$,, then A is invertible and

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{31}\begin{pmatrix} -1 & 19 & -8 \\ -4 & 14 & -1 \\ 8 & 3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{-1}{31} & \frac{19}{31} & \frac{-8}{31} \\ \frac{-4}{31} & \frac{14}{31} & \frac{-1}{31} \\ \frac{8}{31} & \frac{3}{31} & \frac{2}{31} \end{pmatrix}$$

Example 4 Show that $\begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$ is not invertible

Solution $\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = (1)(-6) - (3)(-2) = 0$. Thus, the inverse does not exist.

Theorem 6.2

If A and B are two invertible matrices of the same order, then

$(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

If A and B are invertible matrices of the same order, then $|A| \neq 0$ and $|B| \neq 0$.

$\Rightarrow |AB| = |A||B| \neq 0$

Hence, AB is invertible with inverse $(AB)^{-1}$. On the other hand,

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I$ and similarly

$(B^{-1}A^{-1})(AB) = I$.

Therefore $B^{-1}A^{-1}$ is an inverse of AB and inverse of a matrix is unique.

Hence $B^{-1}A^{-1} = (AB)^{-1}$.

Example 5 Verify that $(AB)^{-1} = B^{-1}A^{-1}$, for the following matrices:

$A = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 2 \\ 3 & 1 \end{pmatrix}$

Solution $|A| = 2$ and $|B| = -9$. To find adj(A), interchange the diagonal elements and take the negatives of the non-diagonal elements. Thus,

$$\text{adj}(A) = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix} \text{ and } \text{adj}(B) = \begin{pmatrix} 1 & -2 \\ -3 & -3 \end{pmatrix}$$

It follows that, $A^{-1} = \frac{1}{|A|}\text{adj}\,A = \frac{1}{2}\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{pmatrix}$, while

$$B^{-1} = \frac{1}{|B|}\text{adj}\,B = -\frac{1}{9}\begin{pmatrix} 1 & -2 \\ -3 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

This gives us $B^{-1}A^{-1} = \begin{pmatrix} -\frac{1}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -1 \\ \frac{-5}{2} & 2 \end{pmatrix} = \begin{pmatrix} -\frac{13}{18} & \frac{5}{9} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

On the other hand, $AB = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 10 \\ -6 & 13 \end{pmatrix}$, so that

$$|AB| = -18 \text{ and } \text{adj}(AB) = \begin{pmatrix} 13 & -10 \\ 6 & -6 \end{pmatrix}.$$

$$(AB)^{-1} = -\frac{1}{18}\begin{pmatrix} 13 & -10 \\ 6 & -6 \end{pmatrix} = \begin{pmatrix} -\frac{13}{18} & \frac{5}{9} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Therefore, $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise 6.4

1 Show that $\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$ and $\begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$ are inverses of each other.

2 Find the inverse, if it exists, for each of the following matrices:

a $\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$

3 Show that the matrix $A = \begin{pmatrix} 3-k & 6 \\ 2 & 4-k \end{pmatrix}$ is singular when $k = 0$ or $k = 7$. What is the inverse when $k = 1$?

4 Given $A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, show that $A^{-1} = A^T$.

5 Using $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

6 Prove that if A is non-singular, then $AB=AC$ implies $B=C$. Does this necessarily hold if A is singular? If not, try to produce an example to the contrary.

6.4 SYSTEMS OF EQUATIONS WITH TWO OR THREE VARIABLES

Matrices are most useful in solving systems of linear equations. Systems of linear equations are used to give mathematical models of electrical networks, traffic flow and many other real life situations.

Definition 6.13

An equation $a_1x_1 + a_2x_2 + ... + a_nx_n = b$, where $a_1, a_2, ..., a_n, b$ are constants and $x_1, x_2, ..., x_n$ are variables is called a linear equation. If $b = 0$, then the linear equation is said to be **homogeneous**.

A linear system with m equations in n unknowns (variables) $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ .... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases} \quad (*)$$

The system of equations (*) is equivalent to $AX = B$, where

$$A = (a_{ij})_{m\times n},\ X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}.$$

Matrix A is called **the coefficient matrix** of the system and the matrix

$$(A/B)=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$ is called **the augmented matrix** of the system.

Example 1 Which of the following are systems of linear equations?

a $\begin{cases} 5x-23y=6 \\ x+14y=12 \end{cases}$ **b** $\begin{cases} 5x^2-23y=6 \\ x+14y=12 \end{cases}$ **c** $\begin{cases} 5x-23y+z=6 \\ x+14y-4z=18 \end{cases}$

Solution a and c are systems of linear equations. b is not a linear equation because the first equation in the system is not linear in x.

Example 2 Give the augmented matrix of the following systems of equations.

a $\begin{cases} 2x+5y=1 \\ 3x-8y=4 \end{cases}$ **b** $\begin{cases} 2x-y+z=3 \\ 3x-2y+8z=-24 \\ x+3y+4y=-2 \end{cases}$ **c** $\begin{cases} x+y=0 \\ 2x-y+3z=3 \\ x-2y-z=3 \end{cases}$

Solution

a $\begin{pmatrix} 2 & 5 & 1 \\ 3 & -8 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & -1 & 1 & 3 \\ 3 & -2 & 8 & -24 \\ 1 & 3 & 4 & -2 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 3 \\ 1 & -2 & -1 & 3 \end{pmatrix}$

Elementary operations on matrices

ACTIVITY 6.12

Solve each of the following systems of linear equations.

a $\begin{cases} x+y=5 \\ x-y=1 \end{cases}$ **b** $\begin{cases} 2x-y=4 \\ -x+y=-1 \end{cases}$ **c** $\begin{cases} 3x-5y=-5 \\ x+2y=2 \end{cases}$

From Activity 6.12, equations in a and b have the same solution set. You have the following definition for equations having the same solution set.

Definition 6.14

Two systems of linear equations are **equivalent**, if and only if they have exactly the same solution.

To solve systems of linear equations, you may recall, we use either the substitution method or the elimination method. The method of elimination is more systematic than the method of substitution. It can be expressed in matrix form and matrix operations can be done by computers. The method of elimination is based on equivalent systems of equations.

To change a system of equations into an equivalent system, we use any of the following three **elementary** (also called **Gaussian**) **operations.**

Swapping	Interchange two equations of the system.
Rescaling	Multiply an equation of the system by a non-zero constant.
Pivoting	Add a constant multiple of one equation to another equation of the system.

Note:

✓ In the elimination method, the arithmetic involves only the numerical coefficients. Thus it is better to work with the numerical coefficients only.

✓ The numerical coefficients and the constant terms of a system of equations can be expressed in matrix form, called the **augmented matrix,** as shown below in Example 3.

Elementary row operations

Swapping	Interchanging two rows of a matrix
Rescaling	Multiplying a row of a matrix by a non-zero constant
Pivoting	Adding a constant multiple of one row of the matrix onto another row

Elementary column operations

Swapping	Interchanging two columns of a matrix
Rescaling	Multiplying a column of a matrix by a non-zero constant
Pivoting	Adding a constant multiple of one column of the matrix onto another column.

Definition 6.15

Two matrices are said to be row (or column) **equivalent**, if and only if one is obtained from the other by performing any of the elementary operations.

Note:

✓ Since each row of an augmented matrix corresponds to an equation of a system of equations, we will use elementary row operations only.

✓ We shall use the following notations:

- Swapping of i^{th} and j^{th} rows will be denoted by: $R_i \leftrightarrow R_j$
- Rescaling of the i^{th} row by non-zero number r will be denoted by: $R_i \longrightarrow rR_i$
- Pivoting of the i^{th} row by r times the j^{th} row will be denoted by: $R_i \longrightarrow R_i + rR_j$

Example 3 Solve the system of equations given below by using the augmented matrix.

$$\begin{cases} x-2y+z=7 \\ 3x+y-z=2 \\ 2x+3y+2z=7 \end{cases}$$

Solution

Write the augmented matrix	$\begin{pmatrix} 1 & -2 & 1 & 7 \\ 3 & 1 & -1 & 2 \\ 2 & 3 & 2 & 7 \end{pmatrix}$	The objective is to get as many zeros as possible in the coefficients.
$\mathbf{R_2 \longrightarrow R_2 + -3R_1}$	$\begin{pmatrix} 1 & -2 & 1 & 7 \\ 0 & 7 & -4 & -19 \\ 2 & 3 & 2 & 7 \end{pmatrix}$	A zero is obtained in the a_{21} position. Note that the other elements of row 2 are also changed.
$\mathbf{R_3 \longrightarrow R_3 + -2R_1}$	$\begin{pmatrix} 1 & -2 & 1 & 7 \\ 0 & 7 & -4 & -19 \\ 0 & 7 & 0 & -7 \end{pmatrix}$	A zero is obtained in the a_{31} position. Note that the other elements of row 3 are also changed.
$\mathbf{R_3 \longrightarrow R_3 + -1.R_2}$	$\begin{pmatrix} 1 & -2 & 1 & 7 \\ 0 & 7 & -4 & -19 \\ 0 & 0 & 4 & 12 \end{pmatrix}$	A zero is obtained in the a_{32} position. Note that the other elements of row 3 are also changed.

The last matrix corresponds to the system of equation:

$$\begin{cases} x-2y+z=7 \\ 7y-4z=-19 \\ 4z=12 \end{cases}$$

Since this equation and the given equation are equivalent, they have the same solutions. Thus by back- substituting $z = 3$ from the 3rd equation into the 2nd, we get, $y = -1$ and back-substituting $z = 3$ and $y = -1$ in the 1st equation, we get $x = 2$.

The solution set is $\{(2, -1, 3)\}$.

Definition 6.16

A matrix is said to be in **Row Echelon Form** if,

1. a zero row (if there is) comes at the bottom.
2. the first nonzero element in each non-zero row is 1.
3. the number of zeros preceding the first non-zero element in each non-zero row except the first row is greater than the number of such zeros in the preceding row.

Example 4 Which of the following matrices are in echelon form?

$$A=\begin{pmatrix}1 & -2 & -4\\0 & 1 & -2\\0 & 0 & 0\end{pmatrix},\ B=\begin{pmatrix}0 & 0 & 1 & 2\\2 & 3 & 0 & -2\\3 & 3 & 6 & -9\end{pmatrix},\ C=\begin{pmatrix}1 & -2 & 1 & 7\\0 & 7 & -4 & -19\\2 & 3 & 2 & 7\end{pmatrix},\ D=\begin{pmatrix}2 & 3 & 1 & 2\\0 & 0 & 0 & 0\\3 & 3 & -6 & -9\end{pmatrix}$$

Solution

A is in echelon form.

B is not in echelon form because the number of zeros preceding the first non-zero element in the first row is greater than the number of such zeros in the second row. C is not in echelon form for the same reason. D is not in echelon form because the zero row is not at the bottom.

Example 5 Solve the system of equations $\begin{cases} z=2 \\ 2x+3y=-2 \\ 3x+3y+6z=-9 \end{cases}$

Solution

Write the augmented matrix	$\begin{pmatrix}0 & 0 & 1 & 2\\2 & 3 & 0 & -2\\3 & 3 & 6 & -9\end{pmatrix}$	The objective is to get as many zeros as possible in the coefficients.
$\mathbf{R_1 \leftrightarrow R_3}$	$\begin{pmatrix}3 & 3 & 6 & -9\\2 & 3 & 0 & -2\\0 & 0 & 1 & 2\end{pmatrix}$	More zeros moved to last row.
$\mathbf{R_1 \longrightarrow \frac{1}{3}R_1}$	$\begin{pmatrix}1 & 1 & 2 & -3\\2 & 3 & 0 & -2\\0 & 0 & 1 & 2\end{pmatrix}$	A leading entry 1 is obtained in row 1. Note that the other elements of row 1 are also changed.
$\mathbf{R_2 \longrightarrow R_2 + -2R_1}$	$\begin{pmatrix}1 & 1 & 2 & -3\\0 & 1 & -4 & 4\\0 & 0 & 1 & 2\end{pmatrix}$	A zero is obtained at the a_{21} position. Note that the other elements of row 2 are also changed.
$\mathbf{R_1 \longrightarrow R_1 + -1R_2}$	$\begin{pmatrix}1 & 0 & 6 & -7\\0 & 1 & -4 & 4\\0 & 0 & 1 & 2\end{pmatrix}$	A zero is obtained at the a_{12} position. Note that the other elements of row 1 are also changed.

The last matrix corresponds to the system of equation:

$$\begin{cases} x+6z=-7 \\ y-4z=4 \\ z=2 \end{cases}$$

Since this last equation and the given equation are equivalent, we get the solution:

$x = -19$, $y = 12$ and $z = 2$.

The solution set is $\{(-19, 12, 2)\}$. The system has exactly one solution.

The last matrix we obtained is said to be in **reduced-echelon form**, as given in the following definition:

Definition 6.17

A matrix is in **Row Reduced Echelon** form, if and only if,

1. it is in echelon form
2. the first non-zero element in each nonzero row is the only non-zero element in its column.

Example 6 Solve the system of equations $\begin{cases} x + 2y = 0 \\ 2x + y = 1 \\ x - y = 2 \end{cases}$

Solution

Augmented matrix	$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$	
$R_2 \longrightarrow R_2 + -2R_1$ $R_3 \longrightarrow R_3 + -1R_1$	$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & -3 & 2 \end{pmatrix}$	
$R_3 \longrightarrow R_3 + -1R_2$	$\begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	
$R_2 \longrightarrow -\frac{1}{3}R_2$	$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & \frac{-1}{3} \\ 0 & 0 & 1 \end{pmatrix}$	Notice that this matrix is in Row Echelon Form.

In the last row, the coefficient entries are 0, while the constant is 1. This means that $0x + 0y = 1$. But, this has no solution.

Thus, $\begin{cases} x + 2y = 0 \\ 2x + y = 1 \\ x - y = 2 \end{cases}$ has no solution.

i.e., The solution set is empty set.

Note:

When the augmented matrix is changed into either echelon form or reduced-echelon form and if the last non-zero row has numerical coefficients which are all zero while having non-zero constant part, then the system has no solution.

Example 7 Solve the following system of equations

$$\begin{cases} x - 2y - 4z = 0 \\ -x + y + 2z = 0 \\ 3x - 3y - 6z = 0 \end{cases}$$

Solution

Augmented matrix	$\begin{pmatrix} 1 & -2 & -4 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & -3 & -6 & 0 \end{pmatrix}$	
$R_2 \longrightarrow R_2 + R_1$ $R_3 \longrightarrow R_3 + -3R_1$	$\begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 6 & 0 \end{pmatrix}$	
$R_2 \longrightarrow -1R_2$	$\begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 6 & 0 \end{pmatrix}$	
$R_3 \longrightarrow R_3 + -3R_2$ $R_1 \longrightarrow R_1 + 2R_2$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	The matrix is now in reduced-echelon form.

The last matrix gives the system $\begin{cases} x = 0 \\ y + 2z = 0 \end{cases}$

This has solution $x = 0$, $y = -2z$.

The solution set is $\{(0, -2z, z) \mid z \text{ a real number}\}$.

Notice that the solution set is infinite.

> **✍Note:**
>
> When the augmented matrix is changed into either echelon form or reduced-echelon form and if the number of non-zero rows is less than the number of variables, then the system has an infinite solutions.
>
> The method of solving a system of linear equations by reducing the augmented matrix of the system into Reduced-Echelon form is called **Gaussian Elimination Method**.

Note that the Examples 3 - 7 above give all the possibilities for solution sets of systems of linear equations.

Case 1: There is **exactly one solution**—such a system of linear equations is called **consistent.**

Case 2: There is **no solution**—such a system of linear equations is called **inconsistent.**

Case 3: There is an **infinite number of solutions**—such a system of linear equations is called **dependent**.

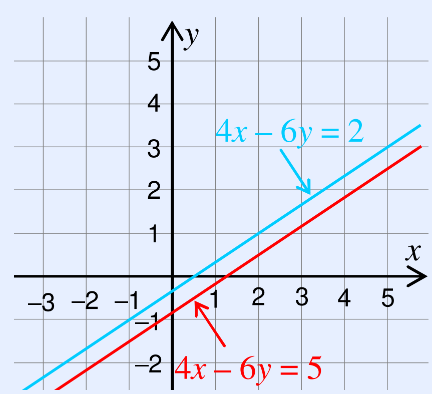
Example 8 Give the solution sets of each of the following system of linear equations. Sketch their graphs and interpret them.

a $\begin{cases} 4x-6y=2 \\ 4x-6y=5 \end{cases}$ **b** $\begin{cases} 5x-4y=6 \\ x+2y=-3 \end{cases}$ **c** $\begin{cases} 3x-y=2 \\ 6x-2y=4 \end{cases}$

Solution

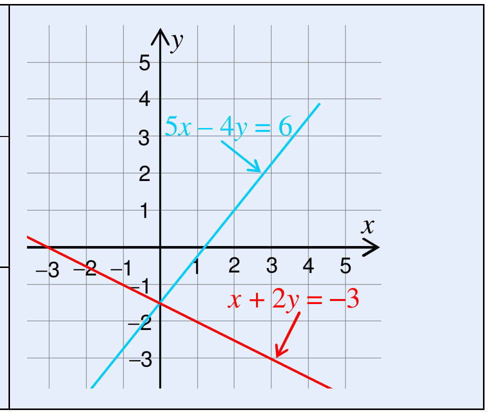
a

Augmented matrix	$\begin{pmatrix} 4 & -6 & 2 \\ 4 & -6 & 5 \end{pmatrix}$
$R_2 \longrightarrow R_2 + -1.R_1$	$\begin{pmatrix} 4 & -6 & 2 \\ 0 & 0 & 3 \end{pmatrix}$



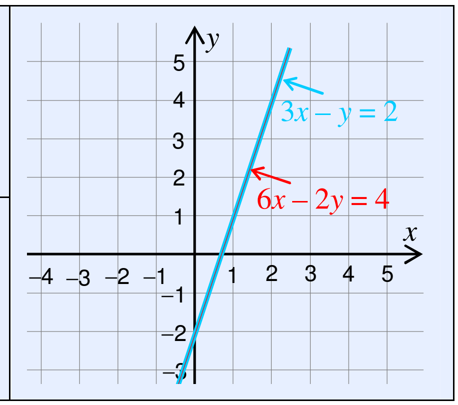
The system has no solution. As you can see from the figure, the two lines are parallel i.e., the two lines do not intersect.

b

Augmented matrix	$\begin{pmatrix} 5 & -4 & 6 \\ 1 & 2 & -3 \end{pmatrix}$	
$R_1 \leftrightarrow R_2$	$\begin{pmatrix} 1 & 2 & -3 \\ 5 & -4 & 6 \end{pmatrix}$	
$R_2 \rightarrow R_2 + \text{-}5R_1$	$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -14 & 21 \end{pmatrix}$	

Here by back-substitution, $y = -\frac{3}{2}$ and $x = 0$. You can see that the lines intersect at exactly one point $\left(0, -\frac{3}{2}\right)$, which is the solution.

c

Augmented matrix	$\begin{pmatrix} 3 & -1 & 2 \\ 6 & -2 & 4 \end{pmatrix}$	
$R_2 \rightarrow R_2 + \text{-}2.R_1$	$\begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$	

The system has infinite solution. In echelon form, there is only one equation, having two variables. In the graph, there is only one line, i.e., both equations represent this same line.

Exercise 6.5

1 State the row operations you would use to locate a zero in the second column of row one.

a $\begin{pmatrix} 5 & 3 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & -1 & 1 & 5 \\ 4 & 8 & 1 & 6 \end{pmatrix}$

2 Reduce each of the following matrices into echelon form.

a $\begin{pmatrix} 5 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & -1 & 1 & 5 \\ 4 & 8 & 1 & 6 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & -1 & 3 & -6 \\ 5 & 3 & -2 & 4 \\ 1 & 3 & 4 & 11 \end{pmatrix}$

3 Reduce each of the following matrices into reduced - echelon form.

a $\begin{pmatrix} 3 & 5 & -1 & -4 \\ 2 & 5 & 4 & -9 \\ -1 & 1 & -2 & 11 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{pmatrix}$

4 **a** Write $\begin{cases} ax+by=e \\ cx+dy=f \end{cases}$ in the form $AX=B$, where

$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, X=\begin{pmatrix} x \\ y \end{pmatrix}$ and $B=\begin{pmatrix} e \\ f \end{pmatrix}$.

b If A is non-singular, show that $X = A^{-1}B$ is the solution.

c Using a and b above, solve $\begin{cases} 2x+3y=4 \\ 5x+4y=17 \end{cases}$

5 Solve each system of equations using augmented matrices.

a $\begin{cases} 2x-2y=12 \\ -2x+3y=10 \end{cases}$ **b** $\begin{cases} 2x-5y=8 \\ 6x+15y=18 \end{cases}$ **c** $\begin{cases} \frac{x}{3}+\frac{3y}{5}=4 \\ \frac{x}{6}-\frac{y}{2}=-3 \end{cases}$

d $\begin{cases} x-3y+z=-1 \\ 2x+y-4z=-1 \\ 6x-7y+8z=7 \end{cases}$ **e** $\begin{cases} 4x+2y+3z=6 \\ 2x+7y=3z \\ -3x-9y+13=-2z \end{cases}$

6 Find the values of c for which this system has an infinite number of solutions.

$\begin{cases} 2x-4y=6 \\ -3x+6y=c \end{cases}$

7 For what values of k does

$\begin{cases} x+2y-3z=5 \\ 2x-y-z=8 \\ kx+y+2z=14 \end{cases}$ have a unique solution?

8 Find the values of c and d for which both the given points lie on the given straight line.

$cx + dy = 2$; (0, 4) and (2, 16)

9 Find a quadratic function $y = ax^2 + bx + c$, that contains the points (1, 9), (4, 6) and (6, 14).

6.5 CRAMER'S RULE

Determinants can be used to solve systems of linear equations with equal number of equations and unknowns.

The method is practicable, when the number of variables is either 2 or 3.

Consider the system $\begin{cases} a_1x + b_1y = c \\ a_2x + b_2y = d \end{cases}$.

$\begin{cases} a_1b_2x + b_1b_2y = b_2c \\ b_1a_2x + b_1b_2y = b_1d \end{cases}$	Multiplying the 1st equation by b_2 and the 2nd equation by b_1.
$(a_1b_2 - b_1a_2)x = b_2c - b_1d$	Subtracting the first equation from the second.
$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} c & b_1 \\ d & b_2 \end{vmatrix}$	Expressing the above equation in determinant notation.

Let $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and $D_x = \begin{vmatrix} c & b_1 \\ d & b_2 \end{vmatrix}$. Then, if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$,

$$x = \frac{\begin{vmatrix} c & b_1 \\ d & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D}. \quad \text{A similar calculation gives: } y = \frac{\begin{vmatrix} a_1 & c \\ a_2 & d \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_y}{D}$$

The method is called **Cramer's rule** for a system with two equations and two unknowns.

Note:

- ✓ D_x and D_y are obtained by replacing the first and second columns by the constant column vector, respectively.
- ✓ Under similar conditions, the rule holds for three unknowns too.

The system of equations $\begin{cases} a_1x + b_1y + c_1z = d \\ a_2x + b_2y + c_2z = e \\ a_3x + b_3y + c_3z = f \end{cases}$ has exactly one solution, provided that the determinant of the coefficient matrix is non-zero. In this case the solution is:

$$x=\frac{\begin{vmatrix} d & b_1 & c_1 \\ e & b_2 & c_2 \\ f & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}=\frac{D_x}{D},\quad y=\frac{\begin{vmatrix} a_1 & d & c_1 \\ a_2 & e & c_2 \\ a_3 & f & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}=\frac{D_y}{D} \text{ and } z=\frac{\begin{vmatrix} a_1 & b_1 & d \\ a_2 & b_2 & e \\ a_3 & b_3 & f \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}=\frac{D_z}{D}$$

Example 1 Use Cramer's rule to find the solution set of $\begin{cases} 3x-4y=2 \\ 7x+7y=3 \end{cases}$

Solution $D=\begin{vmatrix} 3 & -4 \\ 7 & 7 \end{vmatrix}=49\neq 0.$

Thus, by Cramer's Rule, $x=\dfrac{D_x}{D}=\dfrac{\begin{vmatrix} 2 & -4 \\ 3 & 7 \end{vmatrix}}{49}=\dfrac{26}{49}$ and $y=\dfrac{D_y}{D}=\dfrac{\begin{vmatrix} 3 & 2 \\ 7 & 3 \end{vmatrix}}{49}=-\dfrac{5}{49}$

The solution of the system is $x=\dfrac{26}{49}$, $y=-\dfrac{5}{49}$

Example 2 Using Cramer's Rule solve the following system: $\begin{cases} 2x-2y+3z=0 \\ 7y-9z=1 \\ 5x-2y+6z=-2 \end{cases}$

Solution $D=\begin{vmatrix} 2 & -2 & 3 \\ 0 & 7 & -9 \\ 5 & -2 & 6 \end{vmatrix}=33\neq 0.$

Using Cramer's Rule:

$$x=\frac{D_x}{D}=\frac{\begin{vmatrix} 0 & -2 & 3 \\ 1 & 7 & -9 \\ -2 & -2 & 6 \end{vmatrix}}{33}=\frac{4}{11},\qquad y=\frac{D_y}{D}=\frac{\begin{vmatrix} 2 & 0 & 3 \\ 0 & 1 & -9 \\ 5 & -2 & 6 \end{vmatrix}}{33}=-\frac{13}{11}$$

$$z=\frac{D_z}{D}=\frac{\begin{vmatrix} 2 & -2 & 0 \\ 0 & 7 & 1 \\ 5 & -2 & -2 \end{vmatrix}}{33}=-\frac{34}{33}$$

Therefore, the solution of the system is $x=\dfrac{4}{11}$, $y=-\dfrac{13}{11}$, $z=-\dfrac{34}{33}$

Example 3 One solution of the following system is $x = y = z = 0$ (which is known as the trivial solution). Is there any other solution?

$$\begin{cases} 2x - 2y + 3z = 0 \\ \quad 7y - 9z = 0 \\ 5x - 2y + 6z = 0 \end{cases}$$

Solution As shown in the previous example, $D = \begin{vmatrix} 2 & -2 & 3 \\ 0 & 7 & -9 \\ 5 & -2 & 6 \end{vmatrix} = 33 \neq 0.$

Thus, the system has a unique solution. But we already have one solution, namely, $x = 0$, $y = 0$, $z = 0$. So, it is the only solution.

Remark

In the previous sections, you have seen that the determinant of a matrix can be used to find the inverse of a non-singular matrix. Now you will use it in finding the solution set of a system of linear equations when the number of equations and the number of variables are equal.

Consider the linear system (in matrix form), $AX = B$

If $|A| \neq 0$, then A is invertible and $A^{-1}(AX) = A^{-1}B$

$$\Rightarrow (A^{-1}A)X = A^{-1}B$$

$$\Rightarrow IX = A^{-1}B$$

$$\Rightarrow X = A^{-1}B$$

Therefore, the system has a unique solution.

Example 4 Solve the system $\begin{cases} x + y = 7 \\ 2x + 3y = -3 \end{cases}$

Solution The system is equivalent to $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$

The coefficient matrix is $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ with $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1$

$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$ is invertible with inverse $\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

Hence the solution is: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ -3 \end{pmatrix} = \begin{pmatrix} 24 \\ -17 \end{pmatrix}$, i.e. $x = 24$ and $y = -17$

Exercise 6.6

1 Use Cramer's Rule to solve each of the following systems.

a $\begin{cases} -3x+5y=4 \\ 7x+2y=6 \end{cases}$ **b** $\begin{cases} 4x+y=0 \\ x-6y=7 \end{cases}$

c $\begin{cases} 3x+2y-z=5 \\ x-y+3z=-15 \\ 2x+y+7z=-28 \end{cases}$ **d** $\begin{cases} 2x+3y=5 \\ x+3z=6 \\ 5y-z=11 \end{cases}$

2 Use Cramer's Rule to determine whether each of the following homogeneous systems has exactly one solution (namely, the trivial one):

a $\begin{cases} -3x+5y=0 \\ 7x+2y=0 \end{cases}$ **b** $\begin{cases} 3x+2y-z=0 \\ 2x+y+z=0 \\ 5x-2y-z=0 \end{cases}$

Key Terms

adjoint	elementary row operations	scalar matrix
augmented matrix	inconsistent	singular and non-singular matrix
cofactor	inverse	skew-symmetric matrix
column	matrix order	square matrix
consistent	minor	symmetric matrix
dependent	reduced-echelon form	transpose
determinant	row	triangular matrix
diagonal matrix	scalar	zero matrix
echelon form		

Summary

1 A **matrix** is a rectangular array of entries arranged in rows and columns.

2 The size or order of a matrix is written as **rows × columns**.

3 A matrix with only one column is called a **column matrix** (**column vector**).

4 A matrix with only one row is called a **row matrix** (**row vector**).

5 A matrix with the same number of rows and columns is called a **square matrix**.

6 A matrix with all entries 0 is called a **zero matrix**.

7 A **diagonal matrix** is a square matrix that has zeros everywhere except possibly along the main diagonal.

8 The **identity** (**unity**) matrix is a diagonal matrix where all the elements of the diagonal are ones.

9 A **scalar matrix** is a diagonal matrix where all elements of the diagonal are equal.

10 A **lower triangular matrix** is a square matrix whose elements above the main diagonal are all zero.

11 An **upper triangular matrix** is a square matrix whose elements below the main diagonal are all zero.

12 Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ be two matrices. Then,

$A + B = (a_{ij} + b_{ij})_{m\times n}$ and $A - B = (a_{ij} - b_{ij})_{m\times n}$.

13 If r is a scalar and A is a given matrix, then rA is the matrix obtained from A by multiplying each element of A by r.

14 If $A=(a_{ij})$ is an $m \times p$ matrix and $B = (b_{jk})$ is a $p \times n$ matrix, then the product AB is a matrix $C= (C_{ik})$ of order $m \times n$, where

$C_{ik} = a_{ij}\, b_{1j} + a_{i2}\, b_{2j} + a_{i3}b_{3j} + \ldots + a_{ip}b_{pj}$.

15 The **transpose of a matrix** ***A*** is the matrix found by interchanging the rows and columns of A. It is denoted by A^T.

16 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

17 A **minor of** a_{ij}, denoted by M_{ij}, is the determinant that results from the matrix when the row and column that contains a_{ij} are deleted.

18 The **cofactor of** a_{ij} is $(-1)^{i+j}M_{ij}$. Denote the cofactor of a_{ij}, by C_{ij}.

19 Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Then we can expand the determinant along any row i or any column j. Thus we have the formulae:

i^{th} row expansion: $|A| = a_{i1}C_{i1}+a_{i2}C_{i2}+a_{i3}C_{i3}$, for any row i (i = 1,2 or 3). or

j^{th} column expansion: $|A| = a_{1j}C_{1j}+a_{2j}C_{2j}+a_{3j}C_{3j}$, for any column j (j = 1,2 or 3).

20 The **adjoint of a square matrix** $A = (a_{ij})$ is defined as the transpose of the matrix $C = (C_{ij})$ where C_{ij} are the cofactors of the elements a_{ij}. Adjoint of A is denoted by $adj\ A$.

21 When A is invertible or non-singular, then $A^{-1} = \frac{1}{|A|} adj(A)$.

22 ***Elementary Row operations:***

Swapping: Interchanging two rows of a matrix.

Rescaling: Multiplying a row of a matrix by a non-zero constant.

Pivoting: Adding a constant multiple of one row of a matrix on another row.

23 A matrix is in **echelon form**, if and only if

a the leading entry (the first non-zero entry) in each row after the first is to the right of the leading entry in the previous row.

b if there are any rows with no leading entries (rows having zeros entirely) they are at the bottom.

24 A matrix is in **reduced-echelon form**, if and only if

a it is in echelon form

b the leading entry is 1.

c every entry of a column that has a leading entry, is zero (except the leading entry).

25 If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, the solutions of $\begin{cases} a_1x + b_1y = c \\ a_2x + b_2y = d \end{cases}$ are given by

$$x = \frac{\begin{vmatrix} c & b_1 \\ d & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D}, \qquad y = \frac{\begin{vmatrix} a_1 & c \\ a_2 & d \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_y}{D}.$$

26 If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then the solutions of $\begin{cases} a_1x + b_1y + c_1z = d \\ a_2x + b_2y + c_2z = e \\ a_3x + b_3y + c_3z = f \end{cases}$ are

$$x = \frac{\begin{vmatrix} d & b_1 & c_1 \\ e & b_2 & c_2 \\ f & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_x}{D}, \quad y = \frac{\begin{vmatrix} a_1 & d & c_1 \\ a_2 & e & c_2 \\ a_3 & f & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_y}{D} \text{ and } z = \frac{\begin{vmatrix} a_1 & b_1 & d \\ a_2 & b_2 & e \\ a_3 & b_3 & f \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_z}{D}.$$

Review Exercises on Unit 6

1 If $\begin{pmatrix} a & 6 \\ 10 & d \\ e & 0 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 10 & -1 \\ 3 & 0 \end{pmatrix}$, find a, d, and e.

2 If $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 4 & 6 \\ 5 & 8 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 5 \\ 5 & 3 & 2 \\ 0 & 4 & 7 \end{pmatrix}$, find $5A-2B$.

3 Given $A = \begin{pmatrix} 3 & 3 & 5 \\ 0 & -1 & 2 \\ 4 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 5 \\ 2 & -3 \\ 0 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 4 & 5 \\ -2 & 0 \end{pmatrix}$, $X = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$, find where possible:

a AB **b** BA **c** BC **d** CB **e** CX

f X^TCC^T **g** B^TA-2B **h** X^TX **i** B^TB+4C

4 Sofia sells canned food produced by four different producers *A, B, C* and *D*. Her monthly order is:

	A	B	C	D
Beef Meat	300	400	500	600
Tomato	500	400	700	750
Soya Beans	400	400	600	500

Find her order, to the nearest whole number, if

a she increases her total order by 25%.

b she decreases her order by 15%.

5 Kelecha wants to buy 1 hammer, 1 saw and 2 kg of nails, while Alemu wants to buy 1 hammer, 2 saws and 3 kg of nails. They went to two hardware shops and learned the prices in Birr to be:

	Hammer	Saw	Nails
Shop 1	30	35	7
Shop 2	28	37	6

a Write the items matrix I as a 3×2 matrix.

b Write the prices matrix P as a 2×3 matrix.

c Find PI.

d What are Kelecha's cost at shop1 & Alemu's cost at shop 2?

e Should they buy from shop 1 or shop 2?

6 If $\begin{pmatrix} 0 & -3 & -4 \\ m & 0 & 8 \\ 4 & -8 & 0 \end{pmatrix}$ is a skew-symmetric matrix, what is the value of m?

7 **a** For any square matrix A, check that $\frac{A + A^T}{2}$ is symmetric, while $\frac{A - A^T}{2}$ is skew-symmetric.

b Using **a** above, show that any square matrix A is expressible as the sum of a symmetric matrix and a skew-symmetric matrix.

8 Compute the determinants of each of the following matrices

a $\begin{pmatrix} 4 & 3.5 \\ -7 & -20 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 1 & 4 \\ -7 & 0 & 5 \\ -2 & 5 & 8 \end{pmatrix}$

9 If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $\det(rA) = r^2.\det(A)$.

10 Prove that $\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} = 4abc$

11 In each of the following, find x, if

a $\begin{vmatrix} 3x & -1 \\ x & -3 \end{vmatrix} = \frac{3}{2}$ **b** $\begin{vmatrix} -3 & -x \\ 3x & 4 \end{vmatrix} = 15$

12 Find the inverse of the following matrix: $\begin{pmatrix} 2 & 4 & 2 \\ 3 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

13 Reduce the matrix $A=\begin{pmatrix}0 & -1 & 5\\ 1 & 3 & -2\\ 2 & 1 & 4\end{pmatrix}$ to reduced-echelon form.

14 Determine the values of a and b for which the system

$$\begin{cases}3x & -ay=1\\ bx & +4y=6\end{cases}$$

a has only one solution;

b has no solution;

c has infinitely many solutions.

15 Determine the values of a and b for which the system

$$\begin{cases}3x-2y+z=b\\ 5x-8y+9z=3\\ 2x+y+az=-1\end{cases}$$

a has only one solution;

b has infinitely many solutions;

c has no solution.

16 For what values of k does the following system of equations have no solution?

$$\begin{cases}x+2y-z=12\\ 2x-y-2z=2\\ x-3y+kz=11\end{cases}.$$

17 Solve each of the following.

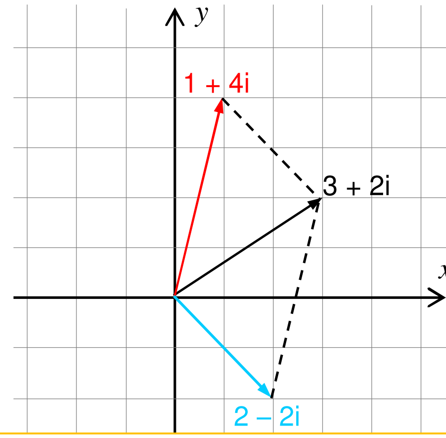
a $\begin{pmatrix}5 & 2 & 1\\ 3 & 2 & 0\\ 1 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}8\\ 5\\ 3\end{pmatrix}$ **b** $\begin{pmatrix}2+\beta & -\beta\\ -\beta & 1+\beta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}5\\ 0\end{pmatrix}$

18 Use Cramer's Rule to solve each of the following.

a $\begin{cases}2x+y=7\\ 3x-2y=0\end{cases}$ **b** $\begin{cases}-x+4y-z=1\\ 2x-y+z=0\\ x+y+z=1\end{cases}$

19 Solve the above by first finding A^{-1} and then using $X = A^{-1}B$.

Unit 7



THE SET OF COMPLEX NUMBERS

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about complex numbers.*
- *know general principles of performing operations on complex numbers.*
- *understand facts and procedures in simplifying complex numbers.*
- *show the geometric representation of complex numbers on the Argand plane.*

Main Contents

INTRODUCTION

Why do we need to study complex numbers?

Why do we need new numbers?

Before introducing complex numbers, let us look at simple examples that illustrate why we need new types of numbers.

For most people, "number" initially meant the whole numbers, 0, 1, 2, 3, ... Whole numbers give us a way to answer questions of the form "How many…?" But whole numbers can answer only some such questions. For example, as you learned to add and subtract, you probably found some subtraction problems, such as 3−5, that you couldn't answer with whole numbers. Furthermore, you probably encountered real-life situations, such as issues of temperature and temperature scales that defied whole-number answers. They showed you that such problems exist in real life as well as in the classroom, and that they need real answers.

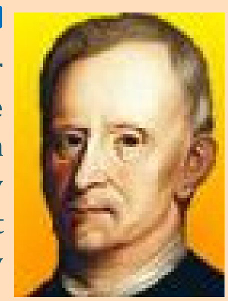
Then you found that if you could work with integers, …, −3, −2, −1, 0, 1, 2, 3, …, all subtraction problems had answers! Clearly negative numbers are needed in real life.

So, by using integers, you can answer all subtraction problems. But what if we are dealing with division? Some – in fact *most* – division problems don't have integer answers. For example, 1÷2, 3÷2, 5÷3 and the like can't be answered with integers. So we need new numbers! We then moved to rational numbers to provide answers to those problems.

There is more to this story. For example, some problems require the use of square roots and other operations – but we won't go into that here. The point is that you have expanded your idea of "number" on several occasions, and now you are about to do so again.

HISTORICAL NOTE

Jean-Robert Argand



Argand was born in July 1768. He was a bookkeeper and amateur mathematician, and is remembered for having introduced the geometrical interpretation of the complex numbers as points in the Cartesian plane. His back ground and education are mostly unknown. Since his knowledge of mathematics was self thought and he did not belong to any mathematical organization, he likely pursued mathematics as a hobby rather than a profession.

OPENING PROBLEM

The "problem" that leads to complex numbers concerns solutions of equations such as,

i $x^2 - 1 = 0$ **ii** $x^2 + 1 = 0$

1 Which equation has real roots? (Equation i or ii?) Can you explain?

2 Draw the graph of $y = x^2 - 1$ and $y = x^2 + 1$ using the same coordinate axes and identify the x-intercepts and y-intercepts of each graph.

In Equation i, -1 and 1 are the two real roots but Equation ii has no real root, since there is no real number whose square is negative.

Do you agree with these answers?

3 Do you see any difference between roots and x-intercepts? So if Equation ii is to be given solutions, then, you must create a square root of -1.

7.1 THE CONCEPT OF COMPLEX NUMBERS

In the above problem, for Equation ii to have solutions, you must create a square root of -1.

In general for any quadratic equation of the form $ax^2 + bx + c = 0$ to have solutions, you need a number system in which $\sqrt{b^2 - 4ac}$ is defined for all numbers a, b and c. The number system which you are going to define is called the **complex number system**.

To this end a new number which is called an **"imaginary number"** namely $\sqrt{-1} = i$ (read as **iota**) is introduced.

Example 1 Using the notation introduced above, you have:

a $\sqrt{-4} = \sqrt{(-1)}\sqrt{4} = 2i$ **b** $\sqrt{-25} = \sqrt{(-1)}\sqrt{25} = 5i$

c $\sqrt{-2} = \sqrt{(-1)\times 2} = \sqrt{-1}\sqrt{2} = \sqrt{2}i$

Now you are ready to define complex numbers as follows:

Definition 7.1

A complex number z is an expression which is written in the form $z = x + yi$, for some real numbers x and y, where $i = \sqrt{-1}$; the number x is called **the real part of z** and is denoted by $\text{Re}(z)$ and the number y is called the **imaginary part of z** and is denoted by $\text{Im}(z)$.

NOTATION:

The set of complex numbers denoted by $\mathbb{C}$ is given by

$\mathbb{C} = \{z|z = x + yi$ where x and y are real numbers; and $i = \sqrt{-1}\ \}$

Note that $i = \sqrt{-1} \Rightarrow i^2 = -1$.

Example 2

a For $z = 2 - 5i$, $\text{Re}(z) = 2$ and $\text{Im}(z) = -5$

b For $z = 6 + 4i$, $\text{Re}(z) = 6$ and $\text{Im}(z) = 4$

c For $z = 0 + 2i = 2i$, $\text{Re}(z) = 0$ and $\text{Im}(z) = 2$

d For $z = 0 + 0i = 0$, $\text{Re}(z) = 0$ and $\text{Im}(z) = 0$

e For $z = 4 + 0i = 4$, $\text{Re}(z) = 4$ and $\text{Im}(z) = 0$

Equality of complex numbers

Suppose $z_1 = x + yi$ and $z_2 = a + bi$ are two complex numbers; then we define the equality of z_1 and z_2, written as $z_1 = z_2$, if and only if $x = a$ and $y = b$.

Example 3 If $15 - 3yi = 3x + 12i$, then $3x = 15$ and $-3y = 12$

Thus, $x = 5$ and $y = -4$.

Exercise 7.1

1 Write the following without exponents.

a i^3 **b** i^4 **c** i^7 **d** i^8

e i^{100} **f** i^{101} **g** i^{102} **h** i^{103}

2 Generalize for i^{2n} and i^{2n+1}.

Hint:- *Consider the case when n is odd and when n is even.*

3 Identify the real and imaginary parts of each of the following complex numbers.

a $\frac{3-5i}{7}$ **b** $\sqrt{5} + 2i\sqrt{2}$ **c** 7 **d** 5i

4 Find the value of the unknowns in each of the following equations.

a $x - 3i = 2 + 12yi$ **b** $7 + 2yi = t - 10i$

5 Write each of the following real numbers in the form of $a + bi$ where a and b are real numbers.

a 3 **b** -7 **c** 0 **d** $\sqrt{13}$

6 Given any real number r, is it always possible to express it as $a + bi$ for some real numbers a and b?

7 Can you conclude that any real number is a complex number? Explain.

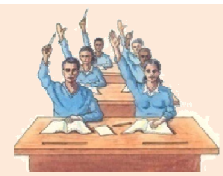
7.2 OPERATIONS ON COMPLEX NUMBERS

From the above exercise and the discussions so far, you can write every real number *r* in the form of $r = r + 0i$; this means that the set of real numbers is a subset of the set of complex numbers. Now the present topic is about extending the operations (addition, subtraction, multiplication and division) on the set of real numbers to the set of complex numbers.

7.2.1 Addition and Subtraction

Before defining addition and subtraction on the set of complex numbers, let us look at your experiences of adding and subtracting terms involving variables as an Activity.

ACTIVITY 7.1



Perform each of the following operations.

a $(2x + 3y) + (5x - 7y)$ **b** $(3x + 4y) - (6x - 2y)$

c $(3 + k) + (5 - 3k)$ **d** $(5 + 4h) - (13 + 2h)$

Now, you have experience in adding expressions such as $(3 - 5x) + (6 + 7x)$. You do it by combining similar terms in the expressions. For example, if you were to simplify the expression $(3 - 5x) + (6 + 7x)$ by combining like terms, then the constants 3 and 6 would be combined to yield 9, and the terms $(-5x)$ and $(7x)$ would be combined to yield $2x$; hence the simplified form is $(9 + 2x)$.

$$\text{i.e., } (3-5x)+(6+7x)=(3+6)+(-5x+7x)=9+2x$$

In a similar fashion, you combine like terms (the real part to the real part and the imaginary part to the imaginary part) in complex numbers when you add or subtract. For instance, given two complex numbers $z_1 = 3 + 4i$ and $z_2 = 5 + 2i$ to find $z_1 + z_2$ you add 3 and 5 together (the real parts) and add 4 and 2 (the imaginary parts) to get $8 + 6i$; and to find $z_1 - z_2$: you subtract 5 from 3 (the real parts) and 2 from 4 (the imaginary parts) to get $-2 + 2i$.

Definition 7.2

Given two complex numbers $z_1 = x + yi$ and $z_2 = a + bi$, we define the sum and difference of complex numbers as follows:

i $z_1 + z_2 = (x + a) + (y + b)i$

ii $z_1 - z_2 = (x - a) + (y - b)i$

Example 1

a $(3 - 5i) + (6 + 7i) = (3 + 6) + (-5 + 7)i = 9 + 2i$

b $(3 - 4i) - (2 + i) = (3 - 2) - (4 + 1)i = 1 - 5i$

Group Work 7.1

1 Given: $z_1 = a + bi$, $z_2 = c + di$ and $z_3 = x + yi$, answer each of the following:

a Is $z_1 + z_2$ a complex number? Explain. What do you call this property?

b Find $z_1 + z_2$ and $z_2 + z_1$. Is $z_1 + z_2 = z_2 + z_1$? What do you call this property?

c Find $z_1 + (z_2 + z_3)$ and $(z_1 + z_2) + z_3$. Is $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$? What do you call this property?

d Find $z_1 + 0, 0 + z_1, (0 = 0 + 0i)$ and compare the values.

Can you conclude that 0 is the additive identity element?

e Find the sums $z + -z$ and $-z + z$.

Can you conclude that $-z$ is the additive inverse of z? Why?

From the above Group Work, you can summarize the following:

- ✓ The set of complex numbers is closed under addition.
- ✓ Addition of complex numbers is commutative.
- ✓ Addition of complex numbers is associative.
- ✓ 0 is the additive identity element in $\mathbb{C}$.
- ✓ For every z in $\mathbb{C}$ there is an additive inverse $-z$ such that $z + -z = 0 = -z + z$.

Exercise 7.2

1 Perform each of the following operations and write your answers in the form of $x + yi$.

a $\sqrt{-9} + \sqrt{-64}$ **b** $(4 + 5i) + (2 - 3i)$

c $(4 + 5i) - (2 - 3i)$ **d** $(7 - 11i) - (3 + 12i)$

e $(2 + \sqrt{-16}) - (1 + \sqrt{-25})$ **f** $i^6 + i^5$

g $i^{12} - i^{16} + i^{21}$ **h** $2i^9 + 3i^{18}$

2 Solve each of the following for x and y.

a $(4 - 2i) + (3 + 5i) = x + yi$ **b** $(10 + 7i) - (2 - 3i) = x + yi$

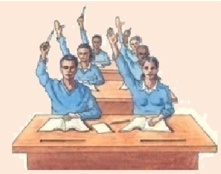
c $(x + yi) + 2(3x - y) + 4i = 0$ **d** $(2x + 3)i + 4(y + 4i) + 5 = 0$

7.2.2 Multiplication and Division of Complex Numbers

Multiplication

Once again, before defining multiplication of complex numbers, let us look at the experience you have in handling multiplication consisting of terms with variables as an Activity.

ACTIVITY 7.2



1 Find each of the following products:

- **a** $(a+b)(a+b)$
- **b** $(a+b)(a-b)$
- **c** $(x+3y)(2x-5y)$
- **d** $(x+3)(x^2+1)$

2 Using the fact $i^2=-1$, find each of the following products:

- **a** $(2+i)(1-i)$
- **b** $(3+2i)(5+17i)$
- **c** $(3+4i)(3-4i)$
- **d** $(3+4i)(3+4i)$

Definition 7.3

Given two complex numbers $z_1=x+yi$ and $z_2=a+bi$, the product of z_1 and z_2 is defined as follows:

$$z_1z_2=(ax-by)+(bx+ay)i$$

You do not need to memorize the formula, because you can arrive at the same result by treating the complex numbers like multiplying terms involving variables; multiply them as usual and then simplify noting that $i^2=-1$.

Example 2 $(2+3i)(4+7i)=2\times4+2\times7i+4\times3i+3i\times7i$

$$=8+14i+12i-21=(8-21)+(14+12)i$$

$$=-13+26i$$

Group Work 7.2

Given, $z_1=a+bi$, $z_2=c+di$ and $z_3=x+yi$; answer the following:

- **a** Is z_1z_2 a complex number? Explain. What do you call this property?
- **b** Is $z_1z_2=z_2z_1$? What do you call this property?
- **c** Is $z_1(z_2z_3)=(z_1z_2)z_3$? What do you call this property?

d Is $z_1(z_2+z_3)=z_1z_2+z_1z_3$? What do you call this property?

e Is $(z_1+z_2)z_3=z_1z_3+z_2z_3$? What do you call this property?

f Find $z_1.1$ and $1.z_1$ $(1=1+0i)$ and compare the values.

Can you conclude that 1 is the multiplicative identity element?

From the above activities you can summarize the following:

- ✓ The set of complex numbers is closed under multiplication.
- ✓ Multiplication of complex numbers is commutative.
- ✓ Multiplication of complex numbers is associative.
- ✓ Multiplication is distributive over addition in $\mathbb{C}$.
- ✓ 1 is the multiplicative identity element in $\mathbb{C}$.

Division

You can think of division as the inverse process of multiplication, since for any two real numbers *a* and *b* with $b \neq 0$ the phrase "*a* is divided by *b*" can be symbolized as:

$$\frac{a}{b}=a\left(\frac{1}{b}\right);\ b\neq 0.$$

Now, do the same thing for complex numbers in the following Group Work.

Group Work 7.3

1 Justify each step in the operation performed below.

$$\left(\frac{1}{2+3i}\right)\left(\frac{1}{2-3i}\right)=\frac{1}{13}$$

$$\frac{1}{2+3i}=\left(\frac{1}{2+3i}\right)\left(\frac{2-3i}{2-3i}\right)$$

$\frac{1}{2+3i}$ is the multiplicative inverse of $2+3i$

$\frac{2}{13}-\frac{3i}{13}$ is the multiplicative inverse of $2+3i$

2 Give reasons for the following arguments.

Given $z=a+bi\neq 0$ $(0=0+0i)$

$$\frac{1}{a+bi}=\left(\frac{1}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right)$$

$$\frac{1}{a+bi}=\frac{a}{a^2+b^2}-\frac{bi}{a^2+b^2}$$

You conclude that $\frac{a}{a^2+b^2}-\frac{bi}{a^2+b^2}$ is the multiplicative inverse of $a+bi$.

Now division of complex numbers can be defined as follows:

Suppose $z_1 = x + yi$ and $z_2 = a + bi \neq 0$ are given, then you have the following:

$$\frac{z_1}{z_2} = z_1 \frac{1}{z_2} = (x+yi)\left(\frac{1}{a+bi}\right) = (x+yi)\left(\frac{a}{a^2+b^2} - \frac{bi}{a^2+b^2}\right)$$

$$= \frac{ax+by}{a^2+b^2} + \frac{(ay-bx)i}{a^2+b^2}$$

Definition 7.4

Suppose $z_1 = x + yi$ and $z_2 = a + bi \neq 0$ are given, then z_1 divided by z_2 denoted by $\frac{z_1}{z_2}$ or $z_1 \div z_2$ is defined to be $z_1 \div z_2 = \frac{z_1}{z_2} = \frac{ax+by}{a^2+b^2} + \frac{(ay-bx)i}{a^2+b^2}$

✍Note:

For every $z \neq 0$ in $\mathbb{C}$ there is its multiplicative inverse $\frac{1}{z}$ such that $z \times \frac{1}{z} = 1 = \frac{1}{z} \times z$.

Example 3

a $\frac{1}{3+7i} = \frac{3}{3^2+7^2} - \frac{7i}{3^2+7^2} = \frac{3}{58} - \frac{7i}{58}$

b $\frac{i+1}{3-4i} = (i+1)\left(\frac{3}{3^2+4^2} - \frac{(-4i)}{3^2+4^2}\right) = (i+1)\left(\frac{3}{25} + \frac{4i}{25}\right)$

$= \frac{-1}{25} + \frac{7i}{25}$

Exercise 7.3

Perform the following operations and write your answers in the form of $a + bi$ where a and b are real numbers.

1 $(-3+4i)(2-2i)$ **2** $3i(2-4i)$ **3** $(2-7i)(3+4i)$

4 $(1+i)(2-3i)$ **5** $(2-i)-i(1-2i)$ **6** $\left(\frac{2-3i}{1-i}\right)\left(\frac{1+i}{2+3i}\right)$

7 $\frac{2-3i}{3+2i} + 6 + 9i$ **8** $i^{12} - i^7$ **9** $i^{20} - i^{24} + i^{15}$

10 $\frac{1}{2+3i}$ **11** $\frac{i+3}{5-2i}$ **12** $\frac{4-2i}{1-i}$

7.3 COMPLEX CONJUGATE AND MODULUS

ACTIVITY 7.3

Given complex numbers $z_1 = x + yi$ and, $z_2 = x - yi$ find

a the product z_1z_2 **b** the sum $z_1 + z_2$ **c** the difference $z_1 - z_2$

From the above activity you can observe the following:

i $(x + yi)(x - yi) = x^2 + y^2$ which is a real number.

ii $(x + yi) + (x - yi) = 2x$ which is twice the real part.

iii $(x + yi) - (x - yi) = 2yi$ which is a purely imaginary number.

The complex number $x - yi$ is called **the conjugate** (or **complex conjugate**) of the complex number $x + yi$. Conjugates are important because of the fact that a complex number multiplied by its conjugate is real; i.e., $(x + yi)(x - yi) = x^2 + y^2$

Definition 7.5

The complex conjugate (or conjugate) of a complex number $z = x + yi$, denoted by $\bar{z}$, is given by $\bar{z} = x - yi$

Example 1

a If $z = 5 - 6i$, then $\bar{z} = 5 - (-6)i = 5 + 6i$

b If $z = -1 + \frac{1}{2}i$, then $\bar{z} = -1 - \frac{1}{2}i$

c If $z = 4 = 4 + 0i$, then $\bar{z} = 4$

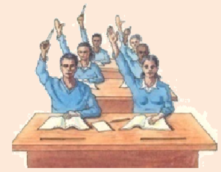
d If $z = -2i$, then $\bar{z} = 2i$

Example 2 In the table below, three columns are filled in; you are expected to fill in the remaining two columns.

Complex number z	Conjugate of z ($\bar{z}$)	Product ($z\bar{z}$)	Sum ($z+\bar{z}$)	Difference ($z-\bar{z}$)
$2 + 3i$	$2 - 3i$	13		
$2 - 3i$	$2 + 3i$	13		
$3 - 5i$	$3 + 5i$	34		
$3 + 5i$	$3 - 5i$	34		
$4i$	$-4i$	16		
$-4i$	$4i$	16		
5	5	25		
$a+bi$	$a-bi$	a^2+b^2		
$a-bi$	$a+bi$	a^2+b^2		

Properties of conjugates

ACTIVITY 7.4



Given two complex numbers $z_1 = 3+4i$ and $z_2 = 5-2i$ find the following:

a $\overline{z}_1$ **b** $\overline{z}_2$ **c** $\overline{z}_1 + \overline{z}_2$

d $z_1 + z_2$ **e** $\overline{z_1 + z_2}$ **f** $\overline{z}_1\overline{z}_2$

g $\overline{z_1 z_2}$ **h** $\dfrac{\overline{z}_1}{\overline{z}_2}$ **i** $\overline{\left(\dfrac{z_1}{z_2}\right)}$

j $\overline{\overline{z}}_1$ **k** $\overline{\overline{z}}_2$

From the above Activity you may summarize properties of conjugates as follows:

Theorem 7.1

For any complex numbers z_1 and z_2, the following properties hold true.

i $\overline{\overline{z}}_1 = z_1$ **ii** $z_1 + \overline{z}_1 = 2\,\text{Re}(z_1)$ **iii** $z_1 - \overline{z}_1 = 2i\,\text{Im}(z_1)$

iv $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$ **v** $\overline{z_1 . z_2} = \overline{z}_1 . \overline{z}_2$ **vi** $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z}_1}{\overline{z}_2}$, if $z_2 \neq 0$

(The proof of this theorem is left as an exercise to you.)

Note that iv and v of the above theorem can be extended to any finite number of terms. i.e.,

$$\overline{z_1 + z_2 + ... + z_n} = \overline{z}_1 + \overline{z}_2 + ... + \overline{z}_n \text{ and } \overline{z_1 . z_2 ..... z_n} = \overline{z}_1 . \overline{z}_2 .... \overline{z}_n$$

One of the important uses of a complex conjugate is to facilitate division of complex numbers. As you have seen, division is the inverse process of multiplication.

i.e., $\dfrac{z_1}{z_2} = z_3$ if and only if $z_1 = z_2 . z_3$ provided $z_2 \neq 0$

If $z_1 = x + yi$, $z_2 = a + bi$ and $z_3 = c + di$, then from $(x+yi)\dfrac{1}{a+bi} = c + di$, one could solve the following:

$$x + yi = (a+bi)(c+di)$$

$$x + yi \cdot \frac{1}{a+bi} = (c+di)$$

$$c = \frac{ax+by}{a^2+b^2} \text{ and } d = \frac{ay-bx}{a^2+b^2} \text{ and conclude that } z_3 = \frac{z_1}{z_2} = \frac{ax+by}{a^2+b^2} + \frac{ay-bx}{a^2+b^2}i$$

However, this is very tedious! Instead, you can use conjugates to simplify expressions of the form $(x + yi) \div (a + bi)$ by writing it in the form of $\frac{x+yi}{a+bi}$ and multiplying both the numerator and denominator by $a - bi$, which is the conjugate of $a + bi$ to arrive at the quotient.

Example 3 If $z_1 = 2+3i$ and $z_2 = 5-i$, then,

$$\frac{z_1}{z_2} = \frac{2+3i}{5-i} = \left(\frac{2+3i}{5-i}\right)\left(\frac{5+i}{5+i}\right) = \frac{7}{26} + \frac{17}{26}i$$

So, one can consider division of a complex number as multiplying both the dividend and the divisor by the conjugate of the divisor.

Definition 7.6

The absolute value (or modulus) of a complex number $z = x + yi$, denoted by $|z|$, is defined to be

$$|z| = \sqrt{x^2 + y^2}$$

This is a natural generalization of the absolute value of real numbers, since $|x+0i| = \sqrt{x^2} = |x|$.

Example 4

a If $z = 2+5i$, then $|z| = \sqrt{2^2+5^2} = \sqrt{29}$

b If $z = 5+12i$, then $|z| = \sqrt{5^2+12^2} = \sqrt{169} = 13$

c If $z = i$, then $|z| = \sqrt{1^2} = 1$

d If $z = -2$, then $|z| = \sqrt{(-2)^2} = |-2| = 2$

✍Note:

If $z_1 = x+yi$ and $z_2 = a+bi$, then

$$|z_1 - z_2| = |(x-a)+(y-b)i| = \sqrt{(x-a)^2+(y-b)^2}$$

Some properties of conjugates and modulus can be summarized as follows:

Theorem 7.2

For any two complex numbers z_1 and z_2 the following properties hold true:

i	$z_1.\bar{z}_1 = \lvert z_1\rvert^2$	**v**	$\lvert z_1.z_2\rvert = \lvert z_1\rvert \cdot \lvert z_2\rvert$
ii	$\lvert z_1\rvert = \lvert \bar{z}_1\rvert$	**vi**	$\left\lvert \frac{z_1}{z_2}\right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$, if $z_2 \neq 0$
iii	$\lvert \mathrm{Re}(z_1)\rvert \leq \lvert z_1\rvert$	**vii**	Triangle inequality: $\lvert z_1 + z_2\rvert \leq \lvert z_1\rvert + \lvert z_2\rvert$
iv	$\lvert \mathrm{Im}(z_1)\rvert \leq \lvert z_1\rvert$	**viii**	$\lvert z_1 - z_2\rvert \geq \lvert z_1\rvert - \lvert z_2\rvert$

Proof:

Let $z_1 = x + yi$ and $z_2 = u + vi$ for some real numbers x, y, u and v

i To show that $z_1.\bar{z}_1 = |z_1|^2$, simply you multiply z_1 with its conjugate $\bar{z}_1 = x - yi$ as follows:

$$z_1 \cdot \bar{z}_1 = (x + yi)(x - yi) = (x^2 + y^2) + (x(-y) + y(x))i = x^2 + y^2 = |z_1|^2$$

ii To show that $|z_1| = |\bar{z}_1|$, since $\bar{z}_1 = x - yi$, you have

$$|\bar{z}_1| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z_1|$$

iii To show that $|\mathrm{Re}(z_1)| \leq |z_1|$, since $x^2 \leq x^2 + y^2$ for every real numbers x and y, you have

$$|\mathrm{Re}(z_1)| = |x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} = |z_1|$$

iv To show that $|\mathrm{Im}(z_1)| \leq |z_1|$, since $y^2 \leq x^2 + y^2$, for every real numbers x and y, you have

$$|\mathrm{Im}(z_1)| = |y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2} = |z_1|$$

v To show that $|z_1.z_2| = |z_1| \cdot |z_2|$,

$$|z_1 \cdot z_2|^2 = (z_1.z_2) \cdot (\overline{z_1.z_2}) \qquad \text{by } \mathbf{i}$$

$$= (z_1.z_2) \cdot (\bar{z}_1.\bar{z}_2) = (z_1 \cdot \bar{z}_1) \cdot (z_2 \cdot \bar{z}_2)$$

$$= |z_1|^2 \cdot |z_2|^2 = (|z_1| \cdot |z_2|)^2$$

$$\Leftrightarrow |z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

vi To show that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, if $z_2 \neq 0$,

$$\left|\frac{z_1}{z_2}\right|^2 = \left(\frac{z_1}{z_2}\right)\cdot\overline{\left(\frac{z_1}{z_2}\right)} = \frac{z_1}{z_2}\cdot\frac{\overline{z}_1}{\overline{z}_2} = \frac{z_1\cdot\overline{z}_1}{z_2\cdot\overline{z}_2} = \frac{|z_1|^2}{|z_2|^2} = \left(\frac{|z_1|}{|z_2|}\right)^2$$

$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, provided that $z_2 \neq 0$

vii To show that $|z_1 + z_2| \leq |z_1| + |z_2|$,

$$|z_1 + z_2|^2 = (z_1 + z_2)\cdot(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2)$$

$$= z_1\cdot\overline{z}_1 + z_1\cdot\overline{z}_2 + z_2\cdot\overline{z}_1 + z_2\cdot\overline{z}_2$$

$$= |z_1|^2 + z_1\cdot\overline{z}_2 + \overline{z_1.\overline{z}_2} + |z_2|^2$$

$$= |z_1|^2 + 2\,\mathrm{Re}\left(z_1\cdot\overline{z}_2\right) + |z_2|^2,$$

But $2\,\mathrm{Re}(z_1.\overline{z}_2) = 2|z_1.\overline{z}_2| = 2|z_1||\overline{z}_2| = 2|z_1||z_2|$.

Thus, $|z_1|^2 + 2\,\mathrm{Re}(z_1 z_2) + |z_2|^2 \leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 = \left(|z_1| + |z_2|\right)^2$

$\Rightarrow |z_1 + z_2|^2 \leq \left(|z_1| + |z_2|\right)^2$

$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|$, which is the required result.

viii To show that $|z_1 - z_2| \geq |z_1| - |z_2|$

$$|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z}_1 - \overline{z}_2) = |z_1|^2 - 2\,\mathrm{Re}(z_1\cdot\overline{z}_2) + |z_2|^2$$

$$\geq |z_1|^2 - 2|z_1||z_2| + |z_2|^2 = \left(|z_1| - |z_2|\right)^2$$

$\Rightarrow |z_1 - z_2| \geq \left||z_1| - |z_2|\right|$

Note:

The triangle inequality can be extended to any finite sum as follows:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

Example 5 Find $|z|$ when $z = \frac{(1+i)^4}{(1+6i)(2-7i)}$

Solution

$$|z| = \frac{|1+i|^4}{|1+6i||2-7i|} = \frac{\sqrt{1^2+1^2}^{\,4}}{\sqrt{1^2+6^2}\sqrt{2^2+(-7)^2}}$$

$$= \frac{\left(\sqrt{2}\right)^4}{\sqrt{37}\sqrt{53}} = \frac{4}{\sqrt{37}\sqrt{53}}$$

Exercise 7.4

1 Perform each of the following operations and write your answers in the form of $a+bi$ where a and b are real numbers.

a $\dfrac{1}{2+3i}$ b $\dfrac{5+4i}{2+3i}$ c $\dfrac{2+3i}{10-4i}$ d $\dfrac{2+i}{3-4i}$

e $\overline{\left(\dfrac{2-3i}{4+5i}\right)}$ f $\dfrac{\overline{1+3i}}{\overline{4-i}}$ g $\dfrac{(2-3i)(4-i)}{(i-1)(i+1)}$ h $\dfrac{(7+i)(3-i)}{2+i}$

2 Given two complex numbers $z_1=3+4i$ and $z_2=6-8i$, find each of the following:

a $|z_1|$ b $|z_2|$ c $|z_1||z_2|$ d $|z_1z_2|$

e Compare the values in c and d.

f $|z_1+z_2|$, $|z_1|+|z_2|$ and compare the two values.

g $|z_1-z_2|$, $|z_1|-|z_2|$ and compare the two values.

h $|z_1|-|z_2|$, $||z_1|-|z_2||$ and compare the two values.

3 Can you conclude that the result noticed in e is true for any two complex numbers $z_1=x+yi$ and $z_2=a+bi$ for real numbers x, y, a, and b?

7.4 SIMPLIFICATION OF COMPLEX NUMBERS

With the help of the concepts discussed so far, you can simplify a given complex expression. Actually simplification means applying the properties of the four operations on a given expression of complex numbers and write it in the form of $a + bi$.

Example 1 Express the following in the form of $a + bi$.

a $\dfrac{(4+2i)(5-6i)}{(1+i)(1-3i)} = \dfrac{20+-24i+10i+12}{1-3i+i+3} = \dfrac{32-14i}{4-2i}$

$$= \left(\frac{32-14i}{4-2i}\right)\left(\frac{4+2i}{4+2i}\right) \text{ (Why?)}$$

$$= \frac{128+64i-56i+28}{16+4} = \frac{156+8i}{20}$$

$$= \frac{39}{5}+\frac{2}{5}i$$

b $\left(1+\sqrt{-81}\right)-(2-\sqrt{-16})+\left(3+\sqrt{196}\right)$

$$= (1+\sqrt{-1}\sqrt{81})-(2-\sqrt{-1}\sqrt{16})+(3+\sqrt{196})$$

$$= (1+9i)-(2-4i)+(3+14) = (1-2+17)+(9i+4i)$$

$$= 16+13i$$

Example 2 Solve (2 – 3i) (x + yi) = 3.

Solution Multiplying both sides of the equation (2 – 3i) (x + yi) = 3 by (2 + 3i) (the complex conjugate) gives;

$$(2+3i)(2-3i)(x+yi)=3(2+3i)\Rightarrow 13(x+yi)=6+9i$$

$$\Rightarrow x+yi=\frac{6}{13}+\frac{9}{13}i\Rightarrow x=\frac{6}{13}\text{ and }y=\frac{9}{13}$$

Example 3 Solve $(x+1)^2=-4$.

Solution $(x+1)^2=-4$

$$\Rightarrow (x+1)=\pm\sqrt{-4}\Rightarrow (x+1)=\pm\sqrt{(-1)\times 4}$$

$$\Rightarrow x+1=\pm 2i\Rightarrow x=-1\pm 2i$$

$$\Rightarrow S.S=\{-1-2i,-1+2i\}$$

An important property of complex numbers is that every complex number has a square root.

Theorem 7.3

If w is a non-zero complex number, then the equation $z^2 = w$ has a solution $z \in \mathbb{C}$.

Proof: Let $w=a+bi,\ a,b\in\mathbb{R}$. You will consider the following two cases.

Case 1 Suppose $b = 0$. Then if $a > 0$, $z=\sqrt{a}$ is a solution, while if $a < 0$, $z=i\sqrt{-a}$ is a solution.

Case 2 Suppose $b \neq 0$. Let $z=x+yi,\ y\in\mathbb{R}$. Then the equation $z^2 = w$ becomes

$$(x+yi)^2 = x^2-y^2+2xyi = a+bi,$$

So equating real and imaginary parts gives

$$x^2-y^2=a \text{ and } 2xy=b$$

Hence, $x \neq 0$ and $y=\dfrac{b}{2x}$

Thus, $x^2 - \left(\frac{b}{2x}\right)^2 = a$

So $4x^4 - 4ax^2 - b^2 = 0$ and $4(x^2)^2 - 4a(x^2) - b^2 = 0$

$$\Rightarrow x^2 = \frac{4a \pm \sqrt{16a^2 + 16b^2}}{8} = \frac{a \pm \sqrt{a^2 + b^2}}{2}$$

Since $x^2 > 0$ you must take the positive sign, as $a - \sqrt{a^2 + b^2} < 0$. Hence

$$x^2 = \frac{a + \sqrt{a^2 + b^2}}{2} \Rightarrow x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}$$

Then, y is determined by $y = \frac{b}{2x}$.

Example 4 Solve the equation $z^2 = 1 + i$.

Solution Put $z = x + yi$ then the equation becomes

$$(x + yi)^2 = x^2 - y^2 + 2xyi = 1 + i$$

$$\Rightarrow x^2 - y^2 = 1 \text{ and } 2xy = 1$$

Hence, $x \neq 0$ and $y = \frac{1}{2x}$. Consequently

$$x^2 - \left(\frac{1}{2x}\right)^2 = 1$$

$$\Rightarrow 4x^4 - 4x^2 - 1 = 0$$

$$\Rightarrow x^2 = \frac{4 \pm \sqrt{16 + 16}}{8} = \frac{1 \pm \sqrt{2}}{2}$$

$$\Rightarrow x = \pm\sqrt{\frac{1 + \sqrt{2}}{2}}$$

Then, $y = \frac{1}{2x} = \pm\frac{1}{\sqrt{2}\sqrt{1 + \sqrt{2}}}$

Hence, the solutions are

$$z = \pm\left(\sqrt{\frac{1 + \sqrt{2}}{2}} + \frac{i}{\sqrt{2}\sqrt{1 + \sqrt{2}}}\right)$$

Example 5 Find the cube roots of 1.

Solution You have to solve the equation $z^3 = 1$, or $z^3 - 1 = 0$

Now, $z^3 - 1 = (z - 1)(z^2 + z + 1)$.

So $z^3 - 1 = 0$ implies $z - 1 = 0$ or $z^2 + z + 1 = 0$

But, $z^2 + z + 1 = 0 \Rightarrow z = \dfrac{-1 \pm \sqrt{1^2 - 4}}{2} = \dfrac{-1 \pm \sqrt{3}i}{2}$

Thus, there are 3 cube roots of 1, namely 1, $\dfrac{-1+\sqrt{3}i}{2}$ and $\dfrac{-1-\sqrt{3}i}{2}$.

Exercise 7.5

1 Write each of the following in the form $a + bi$ where a and b are real numbers.

a $\dfrac{13}{3-2i} - \dfrac{i^3}{1+i}$ **b** $\dfrac{5}{(i-1)(2-i)(3-i)}$ **c** $i^{120} - 4i^{94} + 3i^{31}$

d $\left(2+\sqrt{-25}-\left(3-\sqrt{-216}\right)+\left(1+\sqrt{-9}\right)\right)$ **e** $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$

f $i^{29} + i^{42} + i$ **g** $i^{400} + 3i^{200} + 5i - 3$ **h** $\dfrac{\sqrt{-144}}{\sqrt{-121}}$

i $\left(\sqrt{-12}\right)^3$ **j** $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

2 Given $z_1 = 2 + i$, $z_2 = 3 - 2i$ and $z_3 = \dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}i$, simplify each of the following:

a $z_1^{\,3} - 3z_2^{\,2} + 4z_3$ **b** $\overline{z_3}^{\,4}$ **c** $\left|3\overline{z}_1 - 4\overline{z}_2 + z_3\right|$

d $\dfrac{z_1 z_2}{z_3}$ **e** $\dfrac{z_1 z_3}{z_2}$

3 Solve each of the following equations:

a $z^2 + 4 = 0$ **b** $z^2 + 12 = 0$ **c** $z^2 + z + 1 = 0$

d $3z^2 - 2z + 1 = 0$ **e** $z^3 = -1$ **f** $z^4 = 1$

4 Perform each of the following operations and compare the values obtained:

a $\sqrt{(-4)(-9)}$ **b** $\sqrt{-4}\sqrt{-9}$ **c** $\sqrt{(-4)(9)}$ **d** $\sqrt{-4}\sqrt{9}$

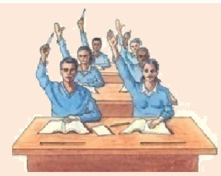
5 If a and b are any real numbers: find conditions for which,

$\sqrt{ab} = \sqrt{a}\sqrt{b}$ and $\sqrt{ab} \neq \sqrt{a}\sqrt{b}$

7.5 ARGAND DIAGRAM AND POLAR REPRESENTATION OF COMPLEX NUMBERS

This sub-unit begins by considering the Cartesian coordinate axes. Previously, you have used a pair of numbers to represent a point in a plane. The main task of this section is to set up a one-to-one correspondence between the set of points in a plane and the set of complex numbers. To this effect let us use the following Activity and group work as a starting point.

ACTIVITY 7.5



1 Consider the set $\{(x, y) \mid x$ and y are real numbers$\}$ in the coordinate plane.

- **a** Locate the two points (2, 3) and (3, 2) in the Cartesian coordinate system. Do they represent the same point or different points? Explain.
- **b** When is $(a, b) = (c, d)$?
- **c** What is the sum (2, 3) + (5, 2)?
- **d** Can you generalize the sum for $(a, b) + (c, d)$?

2 Identify whether each of the following points lie on the x-axis or the y-axis.

- **a** (2, 0)
- **b** $(\frac{1}{2}, 0)$
- **c** $(0, -3)$
- **d** (0.234, 0)
- **e** $(x, 0);\ x \in \mathbb{R}$
- **f** $(0, y);\ y \in \mathbb{R}$

Now you are in a position to set up a one-to-one correspondence between the set of complex numbers and the set of points in a plane, using the correspondence $x + yi \leftrightarrow (x, y)$.

Notation:

The set of points in the plane denoted by $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$ represent the set of all ordered pairs (x, y) of real numbers x and y.

Group Work 7.4

Define a function $f : \mathbb{R}^2 \to \mathbb{C}$ by $f(x, y) = x + iy$ and answer the following:

1 If two points (x, y) and (a, b) with $(x, y) \neq (a, b)$ are given, then is it possible to have $f(x, y) = f(a, b)$? Explain.

2 If a complex number $x + yi$ is given, then does a point (a, b) always exist so that $x + yi = f(a, b)$? Explain.

Geometric representation of complex numbers

The complex number $z = x + yi$ is uniquely determined by the ordered pair of real numbers (x, y). The same is true for the point $P(x, y)$ in the plane with Cartesian coordinates x and y. Hence it is possible to establish a one-to-one correspondence between the set of complex numbers and all points in the plane. You merely associate the complex number $z = x + iy$ with the point $P(x, y)$. The plane whose points represent the complex numbers is called the **complex plane** or the **z-plane**. Real numbers or points corresponding to $x = (x, 0)$ are represented by points on the x-axis; hence the x-axis is called the **Real axis**. Purely imaginary numbers or points corresponding to $iy = (0, y)$ are represented by points on the y-axis, and hence we call the y-axis the **Imaginary axis.** The complex numbers with positive imaginary part lie in the upper half plane, while those with negative imaginary part lie in the lower half plane.

Instead of considering the point $P(x, y)$ as the representation of $z = x + yi$, you may equally consider the directed segment or the vector extending from the origin O to a point P as the representation of a complex number $z = x + yi$. In this case, any parallel segment of the same length and direction is taken as representing the same complex number.

For example, $z = x + yi$, $z_1 = -4 + 2i$ and $z_2 = 2 - 3i$ can be represented as shown in Figure 7.1 below.

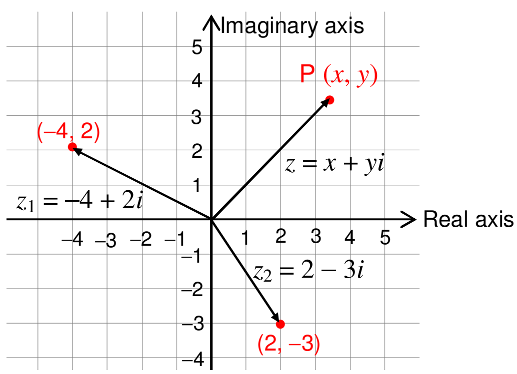


Figure 7.1

$\bar{z}$, $|z|$, and the sum and difference of complex numbers can be presented as follows:

✓ $|z|$ is the length of the vector representing the complex number z or the distance from the origin to the point corresponding to z in the complex plane. More generally, $|z_1 - z_2|$ is the distance between the points corresponding to z_1 and z_2 in the complex plane.

$$|z_1 - z_2| = |(x_1 + y_1 i) - (x_2 + y_2 i)|$$

$$= |(x_1 - x_2) + (y_1 - y_2)i| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

✓ The point corresponding to $\bar{z}$ is the reflection of the point corresponding to z with respect to the real axis.

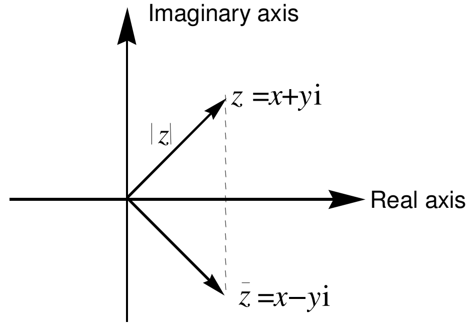


Figure 7.2

Figure 7.2 shows that when the points corresponding to z and $\bar{z}$ are plotted on the complex number plane, one is the reflection of the other.

✓ Because of the equation

$$(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i,$$

complex numbers can be added as vectors using the parallelogram law. Similarly, the complex number $z_1 - z_2$ can be represented by the vector from (x_2, y_2) to (x_1, y_1), where $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$. (See Figure 7.3)

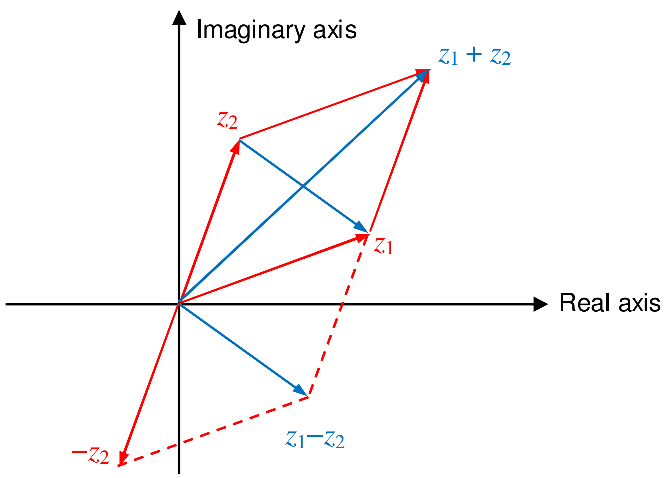


Figure 7.3 *Complex number addition and subtraction*

Polar representation of a complex number

You have seen that a complex number can be represented as a point in the plane. Now, you can use polar coordinates rather than Cartesian coordinates, giving the correspondences (assuming $z \neq 0$)

$$z = x + yi \leftrightarrow (x, y) \leftrightarrow (r, \theta)$$

Let $z = x + yi$ be a non – zero complex number, $r = |z| = \sqrt{x^2 + y^2}$. Then you have $x = r\cos\theta$, $y = r\sin\theta$, where θ is the angle made by the vector corresponding to z with the positive x-axis. (θ is unique upto addition of a multiple of 2π radians.)

From the above discussions, you have:

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

This is called the **polar representation of *z***.

> **Definition 7.7**
>
> When a complex number is written in the form $z = r(\cos\theta + i\sin\theta)$, θ is called an **argument of z** and is denoted by **arg z**. The particular argument of z lying in the range $-\pi < \theta \leq \pi$ is called the **principal argument** of z and is denoted by **Arg z**.

From Figure 7.4, the principal argument of z:

Arg $z = \theta$, you also have arg $z = \theta + 2\pi$

In general,

$r(\cos\theta + i\sin\theta) = r(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi))$, for any integer n, $\theta + 2n\pi$ is also an argument of z, whenever $\theta = \arg(z)$.

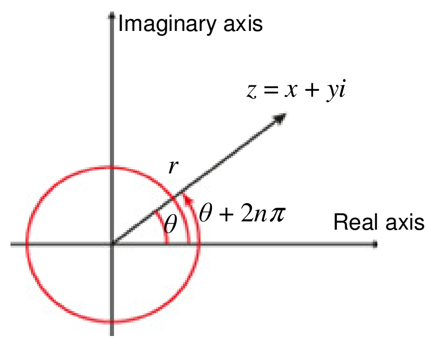


Figure 7.4

Example 1

$$\text{Arg}(1) = 0,\ \text{Arg}(-1) = \pi,\ \text{Arg}(i) = \frac{\pi}{2},\ \text{Arg}(-i) = \frac{-\pi}{2}.$$

Note that $\frac{y}{x} = \tan\theta$ if $x \neq 0$, so θ is determined by this equation up to a multiple of π.

In fact $\text{Arg}\, z = \tan^{-1}\left(\frac{y}{x}\right) + k\pi$,

$$\text{Where} \begin{cases} k = 0, \text{ if } x > 0 \\ k = 1, \text{ if } x < 0,\ y > 0 \\ k = -1, \text{ if } x < 0,\ y < 0 \end{cases}$$

Example 2 Express each of the following complex numbers in polar form.

a $z = 2 + 2\sqrt{3}i$ **b** $z = -5 + 5i$ **c** $z = 3i$ **d** $z = -1$

Solution

a $r = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = \sqrt{4+12} = \sqrt{16} = 4$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3} = \text{Arg}\,(z)$$

Therefore $z = 2 + 2\sqrt{3}i = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ is the polar form of z.

b $r = \sqrt{(-5)^2 + 5^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{5}{-5}\right) = \tan^{-1}(-1) = \frac{3}{4}\pi = \text{Arg}(z)$$

Therefore $z = 5\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$ is the polar form of z.

c $r = \sqrt{0^2 + 3^2} = \sqrt{9} = 3,\ x = 0 \Rightarrow \cos\theta = 0$

$\theta = \cos^{-1}(0) = \frac{4n+1}{2}\pi, n \in Z$. In particular if $n = 0$ then $\theta = \frac{\pi}{2}$.

The principal argument is $\text{Arg}\,(z) = \frac{\pi}{2}$

Therefore, $z = 3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ is the polar form of z.

d $r = \sqrt{(-1)^2 + 0^2} = 1,\ \theta = \sin^{-1}(0)$ and $\theta = \cos^{-1}(-1) \Rightarrow \theta = (2n+1)\pi, n \in Z$

The principal argument: $\text{Arg}\,(z) = \pi$.

Therefore, $z = \cos\pi + i\sin\pi$ is the polar form of z.

Note:

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$z_1 = z_2 \Leftrightarrow r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k, k \in \mathbb{Z}$. (*Why?*)

Example 3

a $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 3\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right) = 3\left(\cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3}\right)$

b $8\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 8\left(\cos\frac{13\pi}{6} + i\sin\frac{13\pi}{6}\right) = 8\left(\cos\frac{11\pi}{6} - i\sin\frac{11\pi}{6}\right)$

The polar representation of a complex number is important because it gives a very simple method of multiplying complex numbers.

Theorem 7.4

Suppose $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then the following hold true.

a $z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

b $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$, provided that $r_2 \neq 0$.

Proof:

a $z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (i\cos\theta_1\sin\theta_2 + i\cos\theta_2\sin\theta_1)\right]$

$= r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$

Hence, a is proved.

The proof of b is left as an exercise to you.

From the above theorem if $\theta_1 = \theta_2 = \theta$ and $r_1 = r_2 = r$ and we have a complex number $z = r(\cos\theta + i\sin\theta)$, then one can show that:

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta);\qquad \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

So one can generalize as follows:

$z^n = r^n(\cos n\theta + i\sin n\theta)$; for any integer n.

Interested students may try the proof for fun!

Remark:

1. If θ is an argument of z, then $n\theta$ is an argument of z^n.
2. If θ is an argument of the non-zero complex number z, then $-\theta$ is an argument of z^{-1}
3. If θ_1 and θ_2 are arguments of z_1 and z_2 then $\theta_1 - \theta_2$ is an argument of $\dfrac{z_1}{z_2}$.
4. In terms of principal argument, you have the following equations:
   - **i** $\text{Arg}(z_1 z_2) = \text{Arg}\, z_1 + \text{Arg}\, z_2 + 2k_1\pi$
   - **ii** $\text{Arg}(z^{-1}) = -\text{Arg}\, z + 2k_2\pi$
   - **iii** $\text{Arg}\left(\dfrac{z_1}{z_2}\right) = \text{Arg}\, z_1 - \text{Arg}\, z_2 + 2k_3\pi$
   - **iv** $\text{Arg}(z_1.....z_n) = \text{Arg}\, z_1 + ... + \text{Arg}\, z_n + 2k_4\pi$
   - **v** $\text{Arg}(z^n) = n\,\text{Arg}(z) + 2k_5\pi$ where k_1, k_2, k_3, k_4, k_5 are integers.
5. It is not always true that $\text{Arg}(z_1 z_2) = \text{Arg}\, z_1 + \text{Arg}\, z_2$

   For example, $\text{Arg}(-1) = \pi$ *but* $\text{Arg}\left|(-1)(-1)\right| = \text{Arg}\ 1 = 0 \neq \pi + \pi$

Example 4 Find the modulus and principal argument of $z=\left(\frac{\sqrt{3}+i}{1+i}\right)^{17}$ and hence express z in polar form.

Solution $|z|=\frac{\left|\sqrt{3}+i\right|^{17}}{|1+i|^{17}}=\frac{2^{17}}{\left(\sqrt{2}\right)^{17}}=2^{\frac{17}{2}}$

$$\text{Arg}\, z = 17\,\text{Arg}\left(\frac{\sqrt{3}+i}{1+i}\right)=17\left(\text{Arg}\left(\sqrt{3}+i\right)-\text{Arg}\,(1+i)\right)$$

$$=17\left(\frac{\pi}{6}-\frac{\pi}{4}\right)=\frac{-17\pi}{12}.$$

Hence $\text{Arg}\, z=\left(\frac{-17\pi}{12}\right)+2k\pi$, where k is an integer. We see that $k = 1$ and hence $\text{Arg}\, z=\frac{7\pi}{12}$.

Consequently, $z=2^{\frac{17}{2}}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$.

Exercise 7.6

1 Give the corresponding representation of the following complex numbers in the Argand plane as points and identify the quadrants to which they belong:

a $1+i$ **b** $2-3i$ **c** $3+4i$ **d** $-1-2i$

2 Express each of the following complex numbers in polar form; and identify the quadrant to which it belongs; find the modulus for each:

a 3 **b** $3i$ **c** -3

d $-3i$ **e** $2+2\sqrt{3}\,i$ **f** $2\sqrt{2}-2\sqrt{2}\,i$

g $-\sqrt{6}-\sqrt{2}\,i$ **h** $\frac{\sqrt{3}}{2}-\frac{3}{2}i$

3 Give the corresponding complex number for each of the following polar representations:

a $(5, \frac{\pi}{3})$ **b** $(6, \frac{\pi}{6})$ **c** $(5, \frac{\pi}{4})$ **d** $(5, 4\frac{\pi}{3})$

4 Find the principal argument for each of the following:

a $z = 4 + 3i$ **b** $z = 4 - 3i$ **c** $z = -2 + 2i$ **d** $z = -2 - 2i$

Key Terms

Argand diagram	**complex plane**	**modulus**
argument	**conjugate**	**polar form**
complex number	**imaginary axis**	**real axis**

Summary

1. An expression of the form $x + yi$ is called a **complex number**, where x and y are real numbers; and $i^2 = -1$. In this expression the number x is called the **real part** of z; and y is called the **imaginary part** of z.
2. A complex number $x - yi$ is called the **conjugate** of a complex number $x + yi$.
3. If $z = x + yi$, then its conjugate denoted by $\bar{z}$ is given by $\bar{z} = x - yi$; the **modulus of** z denoted by $|z|$ is given by $|z| = \sqrt{x^2 + y^2}$.

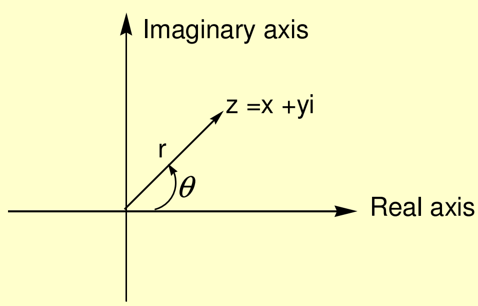


Figure 7.5

4. Let (r, θ) be the polar coordinates of the point representing the complex number $z = x + yi$, $r \geq 0$. Then,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = |z| = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right), \text{ for } x \neq 0$$

$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$ is called the **polar representation** of z.

5. The angle θ is called the **argument** of z and we write it as $\theta = \arg(z)$..
6. Since $r(\cos\theta + i\sin\theta) = r(\cos(\theta + 2\pi n) + i\sin(\theta + 2\pi n))$ for any integer n, then $\theta + 2n\pi$ is also an argument of z for any integer n whenever $\theta = \arg(z)$.

7 Arg(z) is called the **principal argument** of z; it is the value of the argument of z in the interval $(-\pi, \pi]$, that is, $-\pi < \text{Arg}(z) \leq \pi$.

8 If $\text{Arg}(z)$ is the principal argument of z, then $\arg(z) = \text{Arg}(z) + 2\pi n, n \in z$ describes all possible values of $\arg(z)$.

Review Exercises on Unit 7

1 In each of the following solve for x and y.

a $x + yi = i(4 - 3i)$ **b** $\dfrac{1+2i}{x+yi} = 1 - \sqrt{-4}$

c $(3+i)(x + yi)(3 + 4i) = 3 + 9i$ **d** $(2x + yi)(i + 4) = \dfrac{1}{3+5i}$

e $2x + 3xi + 2y = 28 + 9i$

2 Given the complex number $z = 3 + 4i$:

a find the conjugate of z.

b find the modulus of z.

c find the modulus of the conjugate.

d express z in polar form.

3 Find the conjugate, argument and modulus of each of the following expressions.

a $\dfrac{3+i}{5-4i}$ **b** $\dfrac{(2-3i)(4+i)}{\left(i\sqrt{3}+1\right)\left(\frac{1}{2}i+5\right)}$

c $\dfrac{(i+2)(3-4i)(5+3i)}{(2i+1)(4i+3)(5i-3)}$

4 Simplify each of the following and write each in the form of $a + bi$ where a and b are real numbers.

a $i^{320} - 5i^{121} + 3i^{45}$ **b** $\dfrac{1+2i}{6-8i} + \dfrac{6-2i}{10i}$

c $i + (i + 1)^2 + (i - 1)^3 + (i + 2)^4$ **d** $\dfrac{4x}{1-6xi} - \dfrac{2i}{3-i}$

e $\left(\dfrac{1-i}{\sqrt{2}}\right)^{40}$ **f** $(1-i)^{80}$

g $\left(\dfrac{i-\sqrt{3}}{1-i}\right)^{30}$

5 If z_1, z_2 are complex numbers, then prove that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

6 Solve each of the following expressions over $\mathbb{C}$.

a $x^3 + 2x^2 + x - 4 = 0$ **b** $x^2 + 2x + 3 = 0$

c $x^3 - 2x^2 - 3x + 10 = 0$ **d** $x^4 + 2x^2 + 2 = 0$

7 Express each of the following in polar form and evaluate z^{10} for each z.

a $z = 4 + 4\sqrt{3}\,i$ **b** $z = 3\sqrt{2} - 3\sqrt{2}\,i$

c $z = -2\sqrt{6} - 2\sqrt{2}\,i$ **d** $z = \dfrac{\sqrt{3}}{5} - \dfrac{3}{5}i$

e $z = 1 - i\sqrt{3}$ **f** $z = -\sqrt{3} + i$

8 Write the multiplicative inverse for each of the following complex numbers and write the answers in the form of $a + bi$.

a $\dfrac{2+3i}{1+i}$ **b** $\dfrac{5-7i}{2+10i}$ **c** $\dfrac{3+2i}{7-\sqrt{5}i}$

9 Describe each of the following geometrically in a complex plane.

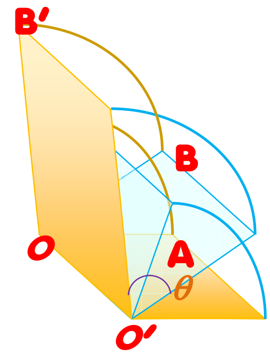
a $|z-1| = 1$ **b** $|z-1| < 1$ **c** $|z-1| > 1$

10 Convert each of the following from polar to Cartesian.

a $\sqrt{2}\left(\cos\dfrac{3\pi}{4} - i\sin\dfrac{3\pi}{4}\right)$ **b** $\sqrt{2}\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$

c $\cos\pi - i\sin\pi$ **d** $5\left(\cos\dfrac{2\pi}{3} - i\sin\dfrac{2\pi}{3}\right)$

Unit 8



VECTORS AND TRANSFORMATION OF THE PLANE

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts and procedures about vectors and operation on vectors.*
- *know specific facts about vectors.*
- *apply principles and theorems about vectors in solving problems involving vectors.*
- *apply methods and procedures in transforming plane figures.*

Main Contents

INTRODUCTION

The measurement of any physical quantity is always expressed in terms of a number and a unit. In physics, for example you come across a number of physical quantities like length, area, mass, volume, time, density, velocity, force, acceleration, momentum, etc. Thus, most of the physical quantities can be divided into two categories as given below.

a Physical quantities having magnitude only

b Quantities having both magnitude and direction

Scalar quantities are completely determined once the magnitude of the quantity is given. However, **vectors** are not completely determined until *both a magnitude and a direction are specified*. For example, wind movement is usually described by giving the speed and the direction, say 20 km/hr northeast. The wind speed and wind direction together form a vector quantity - the wind velocity.

In this unit, you focus on various geometric and algebraic aspects of vector representation and vector algebra.

8.1 REVISION ON VECTORS AND SCALARS

ACTIVITY 8.1

1 Based on your knowledge, classify the measures of the following situations as scalar or vector.

- **a** The width of your classroom.
- **b** The flow of a river.
- **c** The number of students in your class room.
- **d** The direction of your home from your school.
- **e** When an open door is closed.
- **f** When you move nowhere in any direction.

2 Classify each of the following quantities as either vector or scalar:

Displacement, distance, speed, velocity, work, acceleration, area, time,weight, volume, density, force, momentum, temperature, mass.

8.1.1 Vectors and Scalars

In Grade 9, you discussed vectors and their representations. You also discussed vectors and scalars. The following group work and subsequent activities will help you to revise the concepts you learnt.

Group work 8.1

1 Discuss the representation of vectors as coordinate points and as column vectors.

2 Discuss equality of vectors and give examples.

3 When is a vector said to be represented in standard form?

4 If $\mathbf{v}=\begin{pmatrix}3\\4\end{pmatrix}$ is a vector whose initial point is the origin, then find

   a the components of **v**.   b the magnitude of **v**.

   c the direction of **v**.

5 Describe scalar and vector quantities from your surroundings.

Definition 8.1

A quantity which can be completely described by its magnitude expressed in some particular unit is called a **scalar quantity.**

Examples of scalar quantities are mass, time, temperature, etc.

Definition 8.2

A quantity which can be completely described by stating both its magnitude expressed in some particular unit and its direction is called a **vector quantity**.

Examples of vector quantities are velocity, acceleration, etc.

8.1.2 Representation of a Vector

Definition 8.3 Coordinate form of a vector in a plane

If **v** is a vector in the plane whose initial point is the origin and whose terminal point is (x, y), then the coordinate form of **v** is given by $\mathbf{v} = (x, y)$ or $\mathbf{v} = \begin{pmatrix}x\\y\end{pmatrix}$.

The numbers x and y are called **components** (or **coordinates**) of **v**.

Note:

1 If both the initial and terminal points lie at the origin, then **v** is the zero vector and is given by **v** = (0, 0) or $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

2 The above definition implies that two vectors are equal if their corresponding components are equal.

The following procedure can be used to convert directed line segments to coordinate form and vice versa.

1 If P = (x_1, y_1) and Q = (x_2, y_2), are two points on the plane, then the coordinate form of the vector **v** represented by $\overrightarrow{PQ}$ is **v**= ($x_2 - x_1$, $y_2 - y_1$). Moreover, the length of **v** is:

$$|\mathbf{v}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

2 If **v** = (x, y), then **v** can be represented by the directed line segment in standard position, from O = (0, 0) to Q = (x, y).

Example1 Find the coordinate form and the length of the vector **v** that has initial point (3, –7) and terminal point (–2, 5).

Solution Let P = (3, –7) and Q = (–2, 5). Then, the coordinate form of **v** is:

$$\mathbf{v} = (-2 - 3, 5 - (-7)) = (-5, 12)$$

The length of **v** is:

$$|\mathbf{v}| = \sqrt{(-5)^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

Exercise 8.1

Fill in the blank spaces with the appropriate answer.

1 A directed line segment has a _________ and a ________. The magnitude of the directed line segment $\overrightarrow{PQ}$, denoted by __________, is its __________.

2 A vector whose initial point is at the origin **O**(0, 0) can be uniquely represented by the coordinates of its terminal point P(x, y). This is the ___________, written **v** = (x, y), where x and $y$are the _________ of **v**.

3 The coordinate form of the vector with initial point $P = (p_1, p_2)$ and terminal point $Q = (q_1, q_2)$ is $\overrightarrow{PQ}$ = _____________ = **v**.

The magnitude (or length) of **v** is:

$$|\mathbf{v}| = \sqrt{\_\_\_\_\_\_}.$$

4 The coordinate form and magnitude of the vector **v** that has A(1, 7) as its initial point and B(4, 3) as its terminal point are ______________ and ___________.

8.1.3 Addition of Vectors

ACTIVITY 8.2

1 Consider a displacement $\overrightarrow{AB}$ of 3m due N followed by a second displacement $\overrightarrow{BC}$ of 4m due E. Find the combined effect of these two displacements as a single displacement.

2 Consider the following displacement vectors.

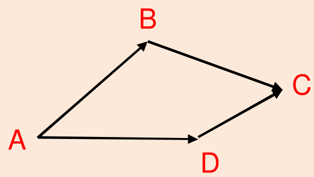


Figure 8.1

Discuss how to determine the combined effect of the vectors as a single vector.

From Activity 8.2 you see that it is possible to add two vectors geometrically using tail to tip rule.

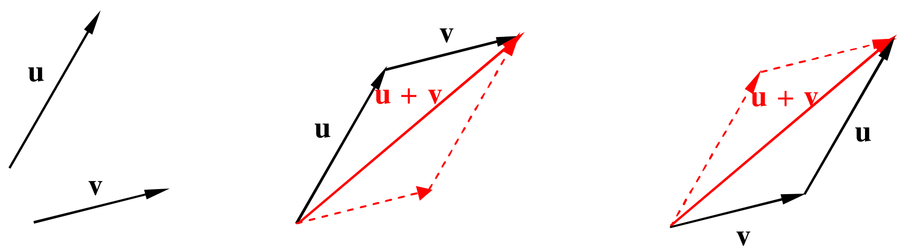


Figure 8.2

To find **u** + **v** *Move the initial point of* **v** *to the terminal point of* **u**. **or** *Move the initial point of* **u** *to the terminal point of* **v**.

Definition 8.4 Addition of vectors (tail-to-tip rule)

If **u** and **v** are any two vectors, the sum **u** + **v** is the vector determined as follows: Translate the vector **v** so that its initial point coincides with the terminal point of **u**. The vector **u** + **v** is represented by the **arrow from the initial point of u to the terminal point of v**.

✍**Note:**

1 One can easily see that **u**, **v** and **u** + **v** are represented by the sides of a triangle, which is called the triangle law of vector addition.

2 The addition of vectors has properties like the real numbers; the two useful properties of vector addition are given below.

Theorem 8.1 Commutative property of vector addition

If **u** and **v** are any two vectors, then

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Proof: Take any point O and draw the vectors $\overrightarrow{OA} = \mathbf{u}$ and $\overrightarrow{AB} = \mathbf{v}$ such that the terminal point of the vector **u** is the initial point of the vector **v** as show in Figure 8.3.

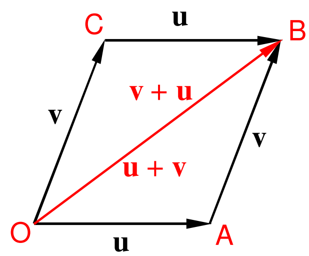


Figure 8.3

Then, by definition of vector addition you have:

$$\mathbf{u} + \mathbf{v} = \overrightarrow{OB} \quad \ldots\ldots\ldots\ldots\ldots\ldots \mathbf{1}$$

Now, completing the parallelogram *OABC* whose adjacent sides are *OA* and *AB*, you infer that $\overrightarrow{OC} = \overrightarrow{AB} = \mathbf{v}$, and $\overrightarrow{CB} = \overrightarrow{OA} = \mathbf{u}$

Using the triangle law of vector addition, you obtain

$$\overrightarrow{OC} + \overrightarrow{CB} = \overrightarrow{OB}$$

$$\mathbf{v} + \mathbf{u} = \overrightarrow{OB} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{2}$$

From 1 and 2, we have:

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Hence, vector addition is *commutative*. This is also called the **parallelogram law of vectors**.

Theorem 8.2 Associative Property of Vector Addition

If **u**, **v**, **w** are any three vectors, then

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

Proof: Let **u**, **v**, **w** be three vectors represented by the line segments as shown in Figure 8.4. i.e., $\mathbf{u} = \overrightarrow{OA},\ \mathbf{v} = \overrightarrow{AB},\ \mathbf{w} = \overrightarrow{BC}$

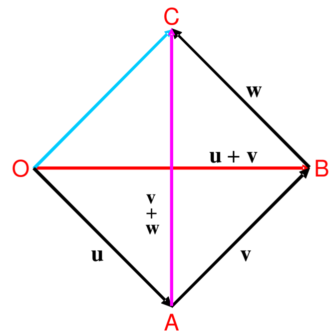


Figure 8.4

Using the definition of vector addition, you have,

i.e., $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{OC} = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$. **1**

Again, you have,

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OA} + (\overrightarrow{AB} + \overrightarrow{BC})$

i.e. $\overrightarrow{OC} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ **2**

Comparing 1 and 2, you have,

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

Hence, vector addition has associative property.

8.1.4 Multiplication of Vectors by Scalars

Group work 8.2

Consider the vector $\overrightarrow{PQ}$

1 What will be $k\overrightarrow{PQ}$, when $k > 0$ and $k < 0$?

2 Discuss the length and direction of $-\overrightarrow{PQ}$. What will be $\overrightarrow{PQ} + (-\overrightarrow{PQ})$?

3 Discuss $\overrightarrow{PQ} + (-\overrightarrow{PQ})$ and $\overrightarrow{PQ} - \overrightarrow{PQ}$

4 If **u** and **v** are two vectors, then represent **u** –**v** geometrically.

Geometrically, the product of a vector **v** and a scalar k is the vector that is k times as long as **v**, as shown in Figure 8.5.

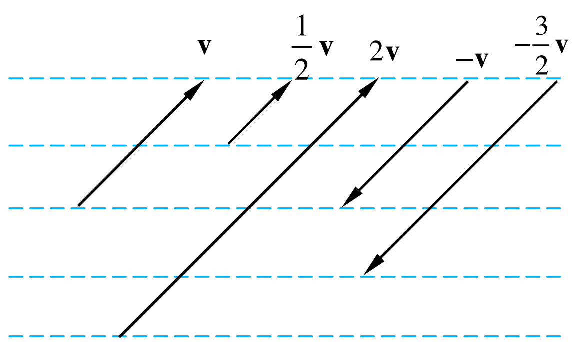


Figure 8.5

If k is positive, then k**v** has the same direction as **v**. If k is negative, then k**v** has the opposite direction.

Example 2 Let **v** be any vector. Then 3**v** is a vector in the same direction as **v** and with length 3 times the length of **v**.

Definition 8.5

If **v** is a non-zero vector and k is a non-zero number (scalar), then the product $k\mathbf{v}$ is defined to be the vector whose length is $|k|$ times the length of **v** and whose direction is the same as that of **v** if $k > 0$ and opposite to that of **v** if $k < 0$.

$k\mathbf{v} = \mathbf{0}$ if $k = 0$ or $\mathbf{v} = \mathbf{0}$.

A vector of the form $k\mathbf{v}$ is called a **scalar multiple** of **v**.

Theorem 8.3

Scalar multiplication satisfies the distributive laws, i.e., if k_1 and k_2 are any two scalars and **u** and **v** are two vectors, then you have:

i $(k_1 + k_2)\,\mathbf{u} = k_1\mathbf{u} + k_2\mathbf{u}$ **ii** $k_1(\mathbf{u} + \mathbf{v}) = k_1\mathbf{u} + k_1\mathbf{v}$

Note:

1. To obtain the difference $\mathbf{u} - \mathbf{v}$ without constructing $-\mathbf{v}$, position **u** and **v** so that their initial points coincide; the vector from the terminal point of **v** to the terminal point of **u** is then the vector $\mathbf{u} - \mathbf{v}$.
2. If **v** is any non-zero vector and $-\mathbf{v}$ is the negative of **v**, then $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
3. For any three vectors **u**, **v** and **w**, if $\mathbf{u} = \mathbf{v}$ and $\mathbf{v} = \mathbf{w}$, then $\mathbf{u} = \mathbf{w}$.
4. The zero vector **0** has the following property: For any vector **u**, $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$.
5. For any vector **u**, $1.\mathbf{u} = \mathbf{u}$
6. If c and d are scalars and **u** is a vector, then $c(d\mathbf{u}) = (cd)\mathbf{u}$.

The operations of vector addition and multiplication by a scalar are easy to work out in terms of coordinate forms of vectors. For the moment, we shall restrict the discussion to vectors in the plane.

Recall from Grade 9 that if $\mathbf{u} = (x_1, y_1)$, $\mathbf{v} = (x_2, y_2)$ and k is a scalar, then

$$\mathbf{u} + \mathbf{v} = (x_1 + x_2, y_1 + y_2);\ k\mathbf{u} = (kx_1, ky_1)$$

Example3 If $\mathbf{u} = (1, -2)$, $\mathbf{v} = (7, 6)$ and $k = 2$, find $\mathbf{u} + \mathbf{v}$ and $2\mathbf{u}$.

Solution $\mathbf{u} + \mathbf{v} = (1 + 7, -2 + 6) = (8, 4)$, $2\mathbf{u} = (2(1), 2(-2)) = (2, -4)$

Definition 8.6

If $\mathbf{u} = (x_1, y_1)$, $\mathbf{v} = (x_2, y_2)$, k is a scalar, then

$$\mathbf{u} + \mathbf{v} = (x_1 + x_2, y_1 + y_2) \qquad k\mathbf{u} = (kx_1, ky_1)$$

Example4 If **u** = (1, –3) and **w** = (4, 2), then **u** + **w** = (5, –1)

2**u** = (2, –6),–**w** = (–4, –2)and **u**– **w** = (–3, –5)

Exercise 8.2

1 A student walks a distance of 3km due east, then another 4km due south. Find the displacement relative to his starting point.

2 A car travels due east at 60km/hr for 15 minutes, then turns and travels at 100km/hr along a freeway heading due north for 15 minutes. Find the displacement from its starting point.

3 Show that if **v** is a non-zero vector and m and n are scalars such that $m\mathbf{v} = n\mathbf{v}$, then $m = n$.

4 Let **u** = (1, 6) and **v** = (–4, 2). Find

a $3\mathbf{u}$ b $3\mathbf{u} + 4\mathbf{v}$ c $\mathbf{u} - \frac{1}{2}\mathbf{v}$

5 What is the resultant of the displacements 6m north, 8m east and 10mnorth west?

6 Draw diagrams to illustrate the following vector equations.

a $\overrightarrow{AB} - \overrightarrow{CB} = \overrightarrow{AC}$ b $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{AD}$

7 If $ABCDEF$ is a regular polygon in which $\overrightarrow{AB}$ represents a vector **v** and $\overrightarrow{BC}$ represents a vector **w,** express each of the following vectors interms of **v** and **w**. $\overrightarrow{CD}, \overrightarrow{DE}, \overrightarrow{EF}$ and $\overrightarrow{AF}$.

8 Using vectors prove that the line segment joining the mid points of the sides of a triangle is half as long as and parallel to the third side.

8.2 REPRESENTATION OF VECTORS

ACTIVITY8.3

1 If **w** is a vector, then discuss how you can express the vector **w** as sum of two other vectors.

2 Using the vectors **i** = (1, 0)and **j** = (0, 1)discuss the following rules of vectors.

i The addition rule: $(a\mathbf{i} + b\mathbf{j}) + (c\mathbf{i} + d\mathbf{j}) = (a + c)\mathbf{i} + (b + d)\mathbf{j}$

ii The subtraction rule: $(a\mathbf{i} + b\mathbf{j}) - (c\mathbf{i} + d\mathbf{j}) = (a - c)\mathbf{i} + (b - d)\mathbf{j}$

iii Multiplication of vectors by scalars: $t(a\mathbf{i} + b\mathbf{j}) = (ta)\mathbf{i} + (tb)\mathbf{j}$

Given a vector **w**, you may want to find two vectors **u** and **v** whose sum is **w**. The vectors **u** and **v** are called **components** of **w** and the process of finding them is called **resolving**, or representing the vector into its vector components.

When you resolve a vector, you generally look for perpendicular components. Most often (in the plane), one component will be parallel to the x-axis and the other will be parallel to the y-axis. For this reason, they are often called the **horizontal** and **vertical** components of a vector.

In the Figure 8.6below, the vector $\mathbf{w} = \overrightarrow{AC}$ is resolved as the sum of $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{BC}$.

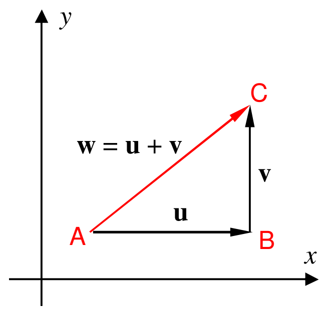


Figure 8.6

The horizontal component of **w** is **u** and the vertical component is **v**.

Example 1 A car weighting 8000N is on a straight road that has a slope of 10° as shown in Figure 8.7. Find the force that keeps the car from rolling down.

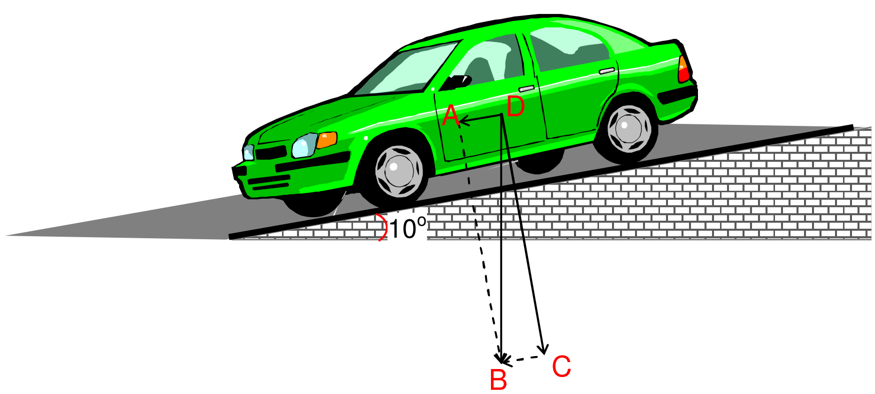


Figure 8.7

Solution The force vector $\overrightarrow{DB}$ acts in the downward direction.

$\Rightarrow \left|\overrightarrow{DB}\right| = 8000\,\text{N}$.

Observe that $\overrightarrow{DC} + \overrightarrow{CB} = \overrightarrow{DB}$ and $m(\angle ABD) = 10^\circ$

$\Rightarrow$ the force that keeps the car at D from rolling down is in the opposite direction of $\overrightarrow{DA}$

$$\Rightarrow \quad \sin(\angle ABD) = \frac{|\overrightarrow{CB}|}{|\overrightarrow{DB}|} = \frac{|\overrightarrow{DA}|}{|\overrightarrow{DB}|} \Rightarrow \sin 10^{o} = \frac{\overrightarrow{DA}}{8000\text{N}}$$

$$\Rightarrow \quad |\overrightarrow{DA}| = 8000 \text{ N} \times \sin 10^{o} = 1389.185 \text{ N}$$

Note:

1 Evidently, a given vector has an infinite number of pairs of possible component vectors. However, if directions of the component vectors are specified, the problem of resolving the vector into component vectors has a unique solution.

2 Let **u** and **v** be two non-zero vectors. In the expression $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$

a the vectors $k_1\mathbf{u}$ and $k_2\mathbf{v}$ are said to be the **components** of **w** relative to **u** and **v**.

b the scalars k_1 and k_2 are called the **coordinates** of the vector **w** relative to **u** and **v.**

Definition 8.7

Two vectors **u** and **v** are said to be **parallel** (**or collinear**), if **u** and **v** lie either on parallel lines or on the same line.

Definition 8.8

Any vector whose magnitude is one is called a **unit vector.**

If **v** is any non-zero vector, the unit vector in the direction of **v** is obtained by multiplying vector **v** by $\frac{1}{|\mathbf{v}|}$. That is, the unit vector in the direction of **v** is $\frac{1}{|\mathbf{v}|}.\mathbf{v}$

The unit vectors (1, 0) and (0, 1) are called the **standard unit vectors in the plane.**

Every pair of non-collinear vectors can be thought of as base. Of course, the components and the coordinates of a given vector in the plane will be different for different bases. For example, the vector **w** = (5, 8) can be written as

(5, 8) = (3, 2) + (2, 6) = (1, 6) + (4, 2) = (5, 0) + (0, 8), etc.

Therefore, (3, 2) and (2, 6), (1, 6) and (4, 2), and (5, 0) and (0, 8),etc are components of **w**.

Your main interest in this section is to find the horizontal and vertical components of a vector **w**, denoted by w_x and w_y.

The unit vectors i and j

Vectors in the xy plane are represented based on the two special vectors **i** = (1, 0) and **j** = (0, 1). Notice that $|\mathrm{i}| = |\mathrm{j}| = 1$. i and j point in the positive directions of the x and y axes, respectively, as shown in Figure 8.8. These vectors are called standard unit base vectors.

Any vector **v** in the plane can be expressed uniquely in the form

$$\mathbf{v} = s\mathbf{i} + t\mathbf{j}$$

where s and t are scalars. In this case, you say that **v** is expressed as a linear combination of **i** and **j**.

Consider a vector **v** whose initial point is the origin and whose terminal point is the point A = (x, y).

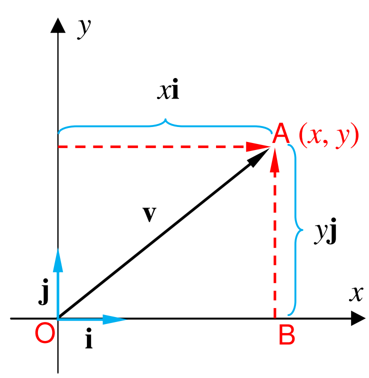


Figure 8.8

If $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\mathrm{i} + y\mathrm{j}$

Note:

The norm of **v**= $|\mathbf{v}| = \sqrt{x^2 + y^2}$

If $\overrightarrow{PQ}$ is a vector with initial point (x_1, y_1) and terminal point (x_2, y_2) as shown in Figure 8.9, then its position vector **v** is determined as

$$\mathbf{v} = (x_2 - x_1, y_2 - y_1) = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$$

Thus, $(x_2 - x_1)$ and $(y_2 - y_1)$ are the coordinates of **v** with respect to the base {**i**, **j**}.

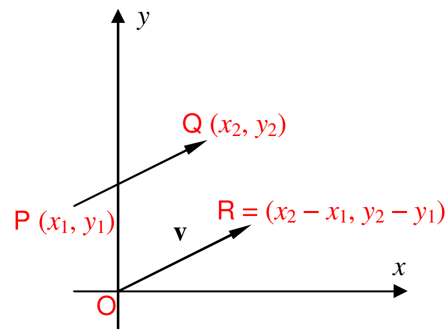


Figure 8.9

Example2 Express the following vectors in terms of the unit vectors **i** and **j** and find their norm.

a (7, –8) **b** (–1, 5) **c** (–2, 3)

Solution

a (7, –8) = 7**i** – 8**j** and its norm (or magnitude) is

$\sqrt{7^2 + (-8)^2} = \sqrt{49 + 64} = \sqrt{113}$

b (–1, 5) = –1**i** + 5**j** and its norm (or magnitude) is

$\sqrt{(-1)^2 + 5^2} = \sqrt{1 + 25} = \sqrt{26}$

c (–2, 3) = –2**i** + 3**j** with norm $\sqrt{13}$

Example 3 Express each of the following as a vector in the coordinate form.

a 3**i**+**j** **b** 2**i**– 2**j** **c** –**i** + 6**j**

Solution

a 3**i** +**j**= 3 (1, 0) + (0, 1) = (3, 0) + (0, 1) = (3, 1)

b 2**i** – 2**j**= 2 (1, 0) – 2 (0, 1) = (2, 0) + (0, –2) = (2, –2)

c –**i** + 6**j**= –(1, 0) + 6 (0, 1) = (–1, 0) + (0, 6) = (–1, 6)

Exercise 8.3

1 Find **u** +**v** for each of the following pairs of vectors

a **u** = (1, 4), **v** = (6, 2) **c** **u** = (2, –2), **v** = (–2, 3)

b **u** = (7, –8), **v** = (–1, 6) **d** **u** = (1 + $\sqrt{2}$, 0), **v** = (–$\sqrt{2}$, 2)

2 Find the norm (or magnitude) of each of the following vectors.

a **u**= (1, 1) **b** **u** = ($\frac{3}{2}$, 0)

c **v** = (–2, 1) **d** **v** = x**i** + y**j** , $x, y \in \mathbb{R}$

3 If $\mathbf{u} = 3\mathbf{i} + \frac{5}{2}\mathbf{j}$ and $\mathbf{v} = \frac{7}{2}\mathbf{i} - \frac{1}{4}\mathbf{j}$, find

a $\mathbf{u} + \mathbf{v}$ **b** $\mathbf{u} - \mathbf{v}$ **c** $t\mathbf{u},\ t \in \mathbb{R}$ **d** $2\mathbf{u} - \mathbf{v}$

4 **a** Find a unit vector in the direction of the vector (2, 4).

b Find a unit vector in the direction opposite to the vector (1, 2).

c Find two unit vectors, one in the same direction as, and the other opposite to the vector $\mathbf{u} = (x, y) \neq 0$.

5 What are the coordinates of the zero vector? Use coordinates to show that

$\mathbf{u} + \mathbf{0} = \mathbf{u}$ for any vector $\mathbf{u}$.

8.3 SCALAR (INNER OR DOT) PRODUCT OF VECTORS

So far you have studied two vector operations, vector addition and multiplication by a scalar, each of which yields another vector. In this section, you will study a third vector operation, **the dot product**. This product yields a scalar, rather than a vector.

Group work 8.3

1 Suppose a body is moved from A to B under a constant force **F** as shown in Figure 8.10. Discuss the uses of **F**, AB and θ.

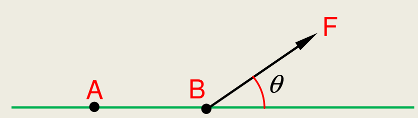


Figure 8.10

2 Let **u** and **v** be two vectors with the same initial point. The angle θ between **u** and **v** is formed as shown in Figure 8.11.

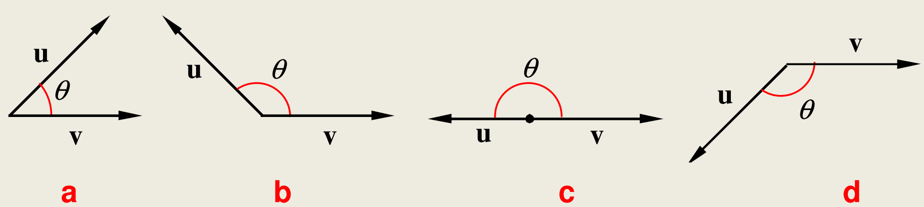


Figure 8.11

Discuss how to express θ in terms of $|\mathbf{u}|$ and $|\mathbf{v}|$.

8.3.1 Scalar (Dot or Inner) Product of Vectors

Definition 8.9

If **u** and **v** are vectors and θ is the angle between **u** and **v**, then the dot product of **u** and **v**, denoted by **u**·**v**, is defined by:

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta.$$

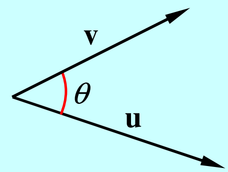


Figure 8.12

Example 1 Find the dot product of the vectors **u** and **v** when

a $\mathbf{u} = (0,\ 1)$ and $\mathbf{v} = (0, 2)$

b $\mathbf{u} = (-2,\ 0)$ and $\mathbf{v} = (\sqrt{3},\ 3)$

Solution Using the definition of dot product, you have

a $|\mathbf{u}| = 1, |\mathbf{v}| = 2$ and $\theta = 0 \Rightarrow \mathbf{u}\cdot\mathbf{v} = 1\times2\times\cos 0^\circ = 2$

b $|\mathbf{u}| = 2, |\mathbf{v}| = \sqrt{\left(\sqrt{3}\right)^2 + 3^2} = 2\sqrt{3}$ and $\theta = 120^\circ$

$$\Rightarrow \mathbf{u}\cdot\mathbf{v} = 2\times2\sqrt{3}\cos 120^\circ = -2\sqrt{3}.$$

Note:

$\mathbf{i}\cdot\mathbf{j} = 0$, $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$

✓ If either **u** or **v** is **0**, then $\mathbf{u}\cdot\mathbf{v} = 0$.

$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ (dot product of vectors is commutative)

✓ If the vectors **u** and **v** are parallel, then $\mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}|\,|\mathbf{v}|$. In particular, for any vector **u**, we have $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$. Here, we write $\mathbf{u}^2$ to mean $|\mathbf{u}|^2$

✓ If the vectors **u** and **v** are perpendicular, then $\mathbf{u}\cdot\mathbf{v} = 0$ because $\cos\left(\frac{\pi}{2}\right) = 0$.

For purposes of computation, it is desirable to have a formula that expresses the dot product of two vectors in terms of the components of the vectors.

In general, using the formula in the definition of the dot product, you can find the angle between two vectors. If **u** and **v** are nonzero vectors, then the cosine of the angle between **u** and **v** is given by:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

The following theorem lists the most important properties of the dot product. They are useful in calculations involving vectors.

Theorem 8.4

Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors and k be a scalar. Then,

i $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(k\mathbf{v})$. . . . . *associative property*

ii $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$ . . . .*distributive property*

$\mathbf{u}\cdot\mathbf{u} > 0$ if $\mathbf{u} \neq \mathbf{0}$, and $\mathbf{u}\cdot\mathbf{u} = 0$ if $\mathbf{u} = \mathbf{0}$

Corollary 8.1

If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors, then $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$.

Proof: $\mathbf{u}\cdot\mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j})\cdot(v_1\mathbf{i} + v_2\mathbf{j})$

$= u_1\mathbf{i}\cdot(v_1\mathbf{i} + v_2\mathbf{j}) + u_2\mathbf{j}\cdot(v_1\mathbf{i} + v_2\mathbf{j})$

$= u_1\mathbf{i}\cdot v_1\mathbf{i} + u_1\mathbf{i}\cdot v_2\mathbf{j} + u_2\mathbf{j}\cdot v_1\mathbf{i} + u_2\mathbf{j}\cdot v_2\mathbf{j}$

$= u_1v_1\mathbf{i}\cdot\mathbf{i} + u_1v_2\,\mathbf{i}\cdot\mathbf{j} + u_2v_1\mathbf{j}\cdot\mathbf{i} + u_2v_2\mathbf{j}\cdot\mathbf{j}$

$= u_1v_1 + u_2v_2$. (Since $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1$ and $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0$)

Example 2 Find the dot product of the vectors $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$

Solution $\mathbf{u}\cdot\mathbf{v} = (3\mathbf{i} + 2\mathbf{j})\cdot(5\mathbf{i} - 3\mathbf{j}) = 3 \times 5 + 2 \times (-3) = 9$

8.3.2 Application of the Dot Product of Vectors

The dot product has many applications. The following are examples of some of them.

Example 3 Find the angle between $3\mathbf{i} + 5\mathbf{j}$ and $-7\mathbf{i} + \mathbf{j}$.

Solution Using vector method,

$(3\mathbf{i} + 5\mathbf{j})\cdot(-7\mathbf{i} + \mathbf{j}) = 3(-7) + 5(1) = 16$

But by definition,

$$(3\mathbf{i} + 5\mathbf{j})\cdot(7\mathbf{i} + \mathbf{j}) = |3\mathrm{i}+5\mathrm{j}|\ |-7\mathrm{i}+\mathrm{j}|\cos\theta = \sqrt{9+25}\sqrt{49+1}\cos\theta$$

$$= \sqrt{34}\sqrt{50}\cos\theta = 16$$

$$\Rightarrow \cos\theta = \frac{16}{\sqrt{34}\sqrt{50}}$$

$$\theta = \cos^{-1}\left(\frac{16}{\sqrt{34}\sqrt{50}}\right)$$

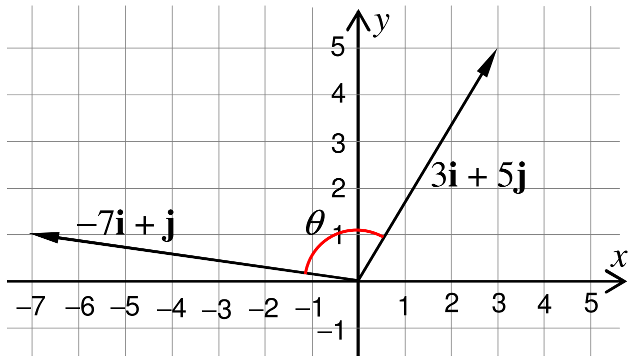


Figure 8.13

The following are some other important properties of the dot product of vectors.

Corollary 8.2

i $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}^2-\mathbf{v}^2$

ii $(\mathbf{u}\pm\mathbf{v})^2 = \mathbf{u}^2 \pm 2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}^2$, where $\mathbf{u}^2=\mathbf{u}\cdot\mathbf{u}$

Example 4 Suppose **a** and **b** are vectors with $|\mathbf{a}| = 4$, $|\mathbf{b}| = 7$ and the angle between **a** and **b** is $\frac{\pi}{3}$.

a Evaluate $|3\mathbf{a}-2\mathbf{b}|$

b Find the cosine of the angle between $3\mathbf{a}-2\mathbf{b}$ and **a**.

Solution Using the properties of dot product we have,

a $|3\mathbf{a}-2\mathbf{b}|^2 = (3\mathbf{a}-2\mathbf{b})\cdot(3\mathbf{a}-2\mathbf{b}) = 9\mathbf{a}^2-12\mathbf{a}\cdot\mathbf{b}+4\mathbf{b}^2$

$$= 9\times16-12|\mathbf{a}||\mathbf{b}|\cos\frac{\pi}{3}+4\times49 = 144-12\times4\times7\times\frac{1}{2}+196$$

$$= 172$$

$$\Rightarrow |3\mathbf{a}-2\mathbf{b}| = \sqrt{172} = 2\sqrt{43}$$

b Let θ be the angle between $3\mathbf{a}-2\mathbf{b}$ and **a**. Then

$$(3\mathbf{a}-2\mathbf{b})\cdot\mathbf{a} = |3\mathbf{a}-2\mathbf{b}||\mathbf{a}|\cos\theta \Rightarrow 3\mathbf{a}^2-2\mathbf{b}\cdot\mathbf{a} = 2\sqrt{43}\times4\cos\theta$$

$$\Rightarrow 3\times16-2|\mathbf{b}||\mathbf{a}|\cos\frac{\pi}{3} = 8\sqrt{43}\cos\theta$$

$$\Rightarrow 48-2\times7\times4\times\frac{1}{2} = 8\sqrt{43}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{5\sqrt{43}}{86}$$

The following statement shows how the dot product can be used to obtain information about the angle between two vectors.

Corollary 8.3

Let **u** and **v** be nonzero vectors. If θ is the angle between them, then

θ is **acute**, if and only if $\mathbf{u}\cdot\mathbf{v}>0$

θ is **obtuse**, if and only if $\mathbf{u}\cdot\mathbf{v}<0$

$\theta=\frac{\pi}{2}$ if and only if $\mathbf{u}\cdot\mathbf{v}=0$

Example 5 Determine the value of k so that the angle between the vectors $\mathbf{u}=(k, 1)$ and $\mathbf{v}=(-2, 3)$ is

a acute **b** obtuse

Solution Using a direct application of Corollary 8.3, we have,

a $\mathbf{u}\cdot\mathbf{v}>0 \Rightarrow (k, 1)\cdot(-2, 3)>0 \Rightarrow -2k+3>0 \Rightarrow k<\frac{3}{2}$

b $\mathbf{u}\cdot\mathbf{v}<0 \Rightarrow k>\frac{3}{2}$.

Observe that the above vectors are perpendicular (orthogonal) if $k=\frac{3}{2}$

Exercise 8.4

1. Find the vectors $\mathbf{z}=\mathbf{u}-2(\mathbf{v}+\mathbf{w})$ and $\mathbf{z}'=(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$, where,
   - **a** $\mathbf{u}=(8, 3)$, $\mathbf{v}=(-1, 2)$, $\mathbf{w}=(1, -4)$
   - **b** $\mathbf{u}=\left(\frac{2}{3},-\frac{1}{2}\right)$, $\mathbf{v}=\left(-3.5,-\frac{4}{5}\right)$, $\mathbf{w}=(-2,-1)$
2. Vectors **u** and **v** make an angle $\theta=\frac{2}{3}\pi$. If $|\mathbf{u}|=3$ and $|\mathbf{v}|=4$, calculate
   - **a** $\mathbf{u}\cdot\mathbf{v}$ **b** $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$ **c** $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$ **d** $|2\mathbf{u}+\mathbf{v}|$
3. Using properties of the scalar product, show that for any vectors **u**, **v**, **w** and **z**,
   - **a** $(\mathbf{u}+\mathbf{v})^2=\mathbf{u}^2+2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}^2$ **b** $(\mathbf{u}-\mathbf{v})^2=\mathbf{u}^2-2\mathbf{u}\cdot\mathbf{v}+\mathbf{v}^2$
   - **c** $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=\mathbf{u}^2-\mathbf{v}^2$ **d** $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{w}+\mathbf{z})=\mathbf{u}\cdot\mathbf{w}+\mathbf{u}\cdot\mathbf{z}+\mathbf{v}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{z}$
4. Let $\mathbf{u}=(1, -1)$, $\mathbf{v}=(1, 1)$ and $\mathbf{w}=(-2, 3)$. Find the cosines of the angles between
   - **a** **u** and **v** **b** **v** and **w** **c** **u** and **w**
5. Prove that if $\mathbf{u}\cdot\mathbf{v}=0$ for all non-zero vectors **v**, then $\mathbf{u}=\mathbf{0}$.
6. Show that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are perpendicular to each other, if and only if $|\mathbf{u}|=|\mathbf{v}|$

7 Show that $(\mathbf{u}\cdot\mathbf{u})\,(\mathbf{v}\cdot\mathbf{v}) \geq (\mathbf{u}\cdot\mathbf{v})^2$. When is $(\mathbf{u}\cdot\mathbf{u})\,(\mathbf{v}\cdot\mathbf{v}) = (\mathbf{u}\cdot\mathbf{v})^2$?

8 **a** Show that $\mathbf{u}\cdot\mathbf{v} = 0 \Leftrightarrow |\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$.

b Consider triangle ABC in Figure 8.14. If the vectors $\mathbf{u} = \overrightarrow{BC}$ and $\mathbf{v} = \overrightarrow{CA}$ are orthogonal, then what is the geometric meaning of the relation in a?

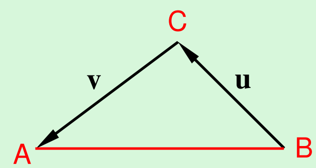


Figure 8.14

9 Vectors $\mathbf{u}$ and $\mathbf{v}$ make an angle $\theta = \dfrac{\pi}{6}$. If $|\mathbf{u}| = \sqrt{3}$ and $|\mathbf{v}| = 1$, then find

a $|\mathbf{u} + \mathbf{v}|$ **b** $|\mathbf{u} - \mathbf{v}|$

10 Let $|\mathbf{u}| = 13$, $|\mathbf{v}| = 19$ and $|\mathbf{u} + \mathbf{v}| = 24$.Calculate

a $\mathbf{u}\cdot\mathbf{v}$ **b** $|\mathbf{u} - \mathbf{v}|$ **c** $|3\mathbf{u} + 4\mathbf{v}|$

8.4 APPLICATION OF VECTOR

From previous knowledge, you notice that vectors have many applications. Geometrically, any two points in the plane determine a straight line. Also a straight line in the plane is completely determined if its slope and a point through which it passes are known. These lines have been determined to have a certain direction. Thus, related to vectors, you will see how one can write equations of lines and circles using vectors.

Example 1 Show that, in a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Solution Let ΔABC be a given right-angled triangle with $\angle C = 90^\circ$.

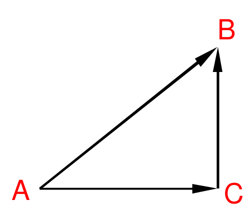


Figure 8.15

Consider the vectors, $\overrightarrow{AC}, \overrightarrow{CB}$ and $\overrightarrow{AB}$ as shown in Figure 8.15.

Since $\angle C = 90^\circ$, $\overrightarrow{AC}.\overrightarrow{CB} = 0$. By vector addition you have $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$. Thus

$$\overrightarrow{AB}^2 = \overrightarrow{AB}.\overrightarrow{AB} = (\overrightarrow{AC}+\overrightarrow{CB}).(\overrightarrow{AC}+\overrightarrow{CB}) = \overrightarrow{AC}^2 + 2\overrightarrow{CB}.\overrightarrow{AC}+\overrightarrow{CB}^2$$

$$= \overrightarrow{AC}^2 + \overrightarrow{CB}^2 \text{ . . . . since } \overrightarrow{CB}.\overrightarrow{AC} = 0$$

Hence, $\overrightarrow{AB}^2 = \overrightarrow{AC}^2 + \overrightarrow{CB}^2$.

Example 2 Show that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent (i.e. they intersect at a single point).

Solution Let ABC be a given triangle and AD and BE be perpendiculars on BC and CA respectively. Suppose AD and BE meet at O as shown in Figure 8.16.

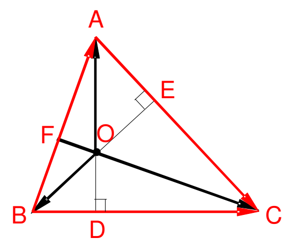


Figure 8.16

Consider the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ and $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$.

Observe that $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}, \overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$ and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

According to our hypothesis, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are perpendicular. Thus

$$\overrightarrow{BC}.\overrightarrow{AD} = 0$$

$$\Rightarrow (\overrightarrow{OC} - \overrightarrow{OB}).\overrightarrow{AD} = 0 \quad \Rightarrow (\overrightarrow{OC} - \overrightarrow{OB}).\overrightarrow{OA} = 0$$

$$\Rightarrow \overrightarrow{OC}.\overrightarrow{OA} = \overrightarrow{OB}.\overrightarrow{OA} \text{ . . . . } \mathbf{1}$$

Similarly, we can write for $\overrightarrow{BE}$ and $\overrightarrow{CA}$, i.e., $\overrightarrow{BE}.\overrightarrow{CA} = 0$

$$\Rightarrow \overrightarrow{BE}.(\overrightarrow{OA} - \overrightarrow{OC}) = 0 \Rightarrow \overrightarrow{OB}.(\overrightarrow{OA} - \overrightarrow{OC}) = 0$$

$$\Rightarrow \overrightarrow{OB}.\overrightarrow{OA} = \overrightarrow{OB}.\overrightarrow{OC} \text{ . . . . } \mathbf{2}$$

By adding 1 and 2, we obtain

$$\overrightarrow{OA}.\overrightarrow{OC} = \overrightarrow{OB}.\overrightarrow{OC} \Rightarrow \overrightarrow{OC}.(\overrightarrow{OB} - \overrightarrow{OA}) = 0 \Rightarrow \overrightarrow{OC}.\overrightarrow{AB} = 0$$

Hence $\overrightarrow{BA}$ and $\overrightarrow{CF}$ are perpendicular.

Thus, the perpendiculars from A, B and C to the opposite sides are concurrent.

Example 3 Prove that the perpendicular bisectors of the sides of a triangle are concurrent.

Solution Let ABC be a triangle and D, E, F the mid-points of BC, CA, and AB, respectively.

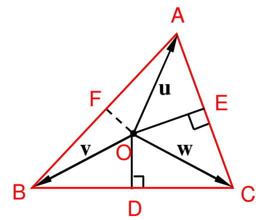


Figure 8.17

DO and EO are perpendiculars to BC and CA respectively. Join O to the mid-point F of AB.

Let **u**, **v**, **w** be the vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$ respectively.

Then, $\overrightarrow{BC} = \mathbf{w} - \mathbf{v}$ and $\overrightarrow{OD} = \dfrac{\mathbf{v}+\mathbf{w}}{2}$

Since $\overrightarrow{OD}$ and $\overrightarrow{BC}$ are perpendicular, you have

$$\overrightarrow{OD}.\overrightarrow{BC} = 0 \text{ i.e. } \left(\frac{\mathbf{v}+\mathbf{w}}{2}\right).(\mathbf{w}-\mathbf{v}) = 0 \ldots\ldots\ldots \mathbf{1}$$

Similarly, since $\overrightarrow{OE}$ and $\overrightarrow{CA}$ are perpendicular, you get

$$\left(\frac{\mathbf{w}+\mathbf{u}}{2}\right).(\mathbf{u}-\mathbf{w}) = 0 \ldots\ldots\ldots\ldots \mathbf{2}$$

From 1 and 2, you obtain $\mathbf{u}^2 - \mathbf{v}^2 = 0$ or $\mathbf{v}^2 - \mathbf{u}^2 = 0$

$\Rightarrow \frac{1}{2}(\mathbf{v}+\mathbf{u}).(\mathbf{v}-\mathbf{u}) = 0 \Rightarrow \overrightarrow{OF}$ and $\overrightarrow{BA}$ are perpendicular.

Apart from the applications discussed above, vectors have many practical applications. Some are presented in the following subunits.

8.4.1 Vectors and Lines

Let $P_o(x_o, y_o)$ and $P_1(x_1, y_1)$ be two points in the plane. Then, the vector from P_o to P_1 is $P_1 - P_o = (x_1 - x_o, y_1 - y_o)$ (*see* Figure8.18).

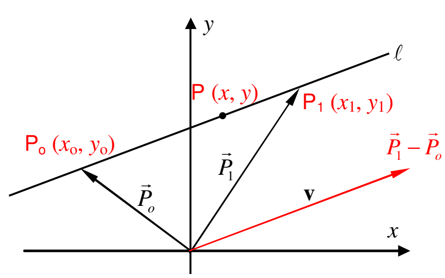


Figure 8.18

$$\vec{P}_1 - \vec{P}_o = (x_1 - x_0,\ y_1 - y_0)$$

where $\vec{P}_o$ and $\vec{P}_1$ are position vectors corresponding to the points P_o and P_1, respectively.

As you can see from Figure 8.18 the line ℓ through P_o and P_1 is parallel to the vector

$\vec{P}_1 - \vec{P}_o = (x_1 - x_o, y_1 - y_o)$.

Let P(x, y) be any point on ℓ. Then the position vector of P is obtained from the relation.

$$\overline{P} - \overline{P}_o = \overline{P_oP} = \lambda\left(P_1 - P_o\right)$$

i.e., $\overline{P} - \overline{P}_o = \lambda(\overline{P}_1 - \overline{P}_o)$, where λ is a scalar.

Observe that you have not used the point P_1 (x_1, y_1) in the above equation except for finding the vector $\mathbf{v} = \overline{P}_1 - \overline{P}_o$, which is often referred to as a **direction** vector of the line. Thus, if a direction vectors **v** and a point $P_o(x_o, y_o)$ are given, then the vector equation of the line determined by P_o and **v** is:

$$P = P_o + \lambda\mathbf{v};\ \lambda \in \mathbb{R}, \mathbf{v} \neq 0.$$

If **v**= (a, b), P (x, y) and P_o (x_o, y_o), then the above equation can be written as:

$$(x, y) = (x_o, y_o) + \lambda\ (a, b)$$

$$or\ \begin{cases} x = x_o + \lambda a \\ y = y_o + \lambda b \end{cases};\ \lambda \in \mathbb{R}, (a,b) \neq (0,0)$$

This system of equations is called the **parametric equation of the line ℓ**, through $P_o(x_o, y_o)$, whose direction is that of the vector $\mathbf{v} = (a, b)$,λ is called a parameter.

Now if a and b are both different from 0, then

$$\frac{x - x_o}{a} = \lambda \text{ and } \frac{y - y_o}{b} = \lambda \Rightarrow \frac{x - x_o}{a} = \frac{y - y_o}{b},$$

which is called the **standard equation of the line**.

The above equation can also be written as:

$$\frac{1}{a}x - \frac{1}{a}x_o = \frac{1}{b}y - \frac{1}{b}y_o \qquad \Rightarrow \frac{1}{a}x - \frac{1}{b}y + \left(\frac{1}{b}y_o - \frac{1}{a}x_o\right) = 0$$

$$\Rightarrow Ax + By + C = 0 \qquad \text{where } A = \frac{1}{a},\ B = -\frac{1}{b} \text{ and } C = \frac{1}{b}y_o - \frac{1}{a}x_o$$

Example 4 Find the vector equation of the line through (1, 3) and (–1, –1)

Solution Here you may take P_o = (1, 3) and P_1 = (–1, –1). Thus, the vector equation of the line is:

$$(x, y) = (1, 3) + \lambda\ ((-1, -1) - (1, 3)) = (1, 3) + \lambda(-2, -4) = (1 - 2\lambda, 3 - 4\lambda)$$

The parametric vector equation is: $x = 1 - 2\lambda \quad y = 3 - 4\lambda, \lambda \in \mathbb{R}$, and

the standard equation is: $\dfrac{x-1}{-2} = \dfrac{y-3}{-4}$

Example 5 Find the vector equations of the line through (1, –2) and with direction vector (3, 1)

Solution You have $P_o = (1, -2)$ and $\mathbf{v} = (3, 1)$. Thus, the vector equation of the line is:

$(x, y) = (1, -2) + \lambda(3, 1) = (1 + 3\lambda, -2+\lambda)$

The parametric vector equation is: $x = 1 + 3\lambda, y = -2 + \lambda, \lambda \in \mathbb{R}$,

The standard equation is given by $\frac{x-1}{3} = \frac{y+2}{1}$

Example6 Find the vector equation of the line passing through the points (2, 3) and (–1,1).

Solution The vector equation of the line passing through two points A and B with position vectors **a** and **b**, respectively, is $P = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$ or $P = \mathbf{b} + \lambda(\mathbf{b} - \mathbf{a})$.

Using this result, $P = (2, 3) + \lambda\ (3, 2)$ or $P = (-1, 1) + \lambda\ (3, 2)$

8.4.2 Vectors and Circles

A circle with centre at $C(x_o, y_o)$ and radius $r > 0$ is the set of all points $P(x, y)$ in the plane such that $\left|\overline{P} - \overline{C}\right| = r$

Where $\overline{P}$ and $\overline{C}$ are position vectors of $P(x, y)$ and $C(x_o, y_o)$ respectively.
(*See* Figure8.19)

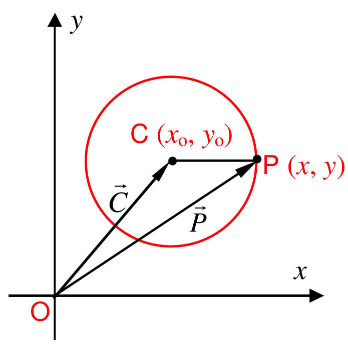


Figure 8.19

By squaring both sides of the equation, we obtain,

$$\left|\overline{P} - \overline{C}\right|^2 = r^2 \quad ....\mathbf{1}$$

$$\left(\overline{P} - \overline{C}\right).\left(\overline{P} - \overline{C}\right) = r^2$$

$$\overline{P}.\overline{P} - 2\overline{P}.\overline{C} + \overline{C}.\overline{C} = r^2 \ ....\ \mathbf{2}$$

The above equation is satisfied by a position vector of any point on the circle. Thus 2 represents the equation of the circle centred at $C(x_o, y_o)$ and radius r.

Substituting the corresponding components of P and C in equation 1, we obtain:

$$(x - x_o)^2 + (y - y_o)^2 = r^2$$

which is called **the standard equation of a circle**.

By expanding and rearranging the terms, this equation can be expressed as:

$$x^2 + y^2 + Ax + By + C = 0, \text{ where } A = -2x_o,\ B = -2y_o \text{ and } C = x_o^2 + y_o^2.$$

Example 7 Find an equation of the circle centred at C(–1, –2) and of radius 2.

Solution Let $P(x, y)$ be a point on the circle.

Let $\overline{P}$ and $\overline{C}$ be the position vectors of P and C, respectively.

Then, from equation (2), we have,

$$(x,\ y).(x,\ y)-2(x,\ y).(-1,-2)+(-1,-2).(-1,-2)=2^2$$

$$\Rightarrow x^2+y^2-2(-x-2y)+(1+4)=4 \Rightarrow x^2+y^2+2x+4y+1=0$$

Example 8 Find the equation of the circle with a diameter the segment from A (5, 3)to B (3, –1).

Solution The centre of the circle is $C(x_o, y_o) = C\left(\frac{5+3}{2}, \frac{3+(-1)}{2}\right) = C\,(4,1)$

The radius of the circle is given by $r=\frac{1}{2}\left(\sqrt{(5-3)^2+(3+1)^2}\right)=\frac{1}{2}\sqrt{4+16}$

$$=\frac{1}{2}\sqrt{20}=\frac{2\sqrt{5}}{2}=\sqrt{5}$$

Let $P(x,\ y)$ be a point on the circle and $\overline{P}$ and $\overline{C}$ be position vectors of P and C respectively. Then, the equation of the circle is:

$$(x,y).(x,\ y)-2(x,\ y).(4,\ 1)+(4,\ 1).(4,\ 1)=\left(\sqrt{5}\right)^2,$$

$$\Rightarrow x^2+y^2-2(4x+y)+16+1=5$$

$$\Rightarrow x^2+y^2-8x-2y+12=0$$

8.4.3 Tangent Line to a Circle

A line tangent to a circle is characterized by the fact that the radius at the point of tangency is perpendicular (orthogonal) to the line.

Let the circle be given by

$$(x-x_o)^2+(y-y_o)^2=r^2,\ r>0$$

Let ℓ be the line tangent to the circle at $P_1\ (x_1, y_1)$.

If $P(x, y)$ is an arbitrary point on ℓ, $\overrightarrow{P_1C}.\overrightarrow{P_1P}=0$

Therefore, the equation of the tangent line must be:

$$(x-x_1,\ y-y_1).(x_1-x_o,\ y_1-y_o)=0$$

$$\Rightarrow \quad (x-x_1)(x_1-x_o)+(y-y_1)(y_1-y_o)=0$$

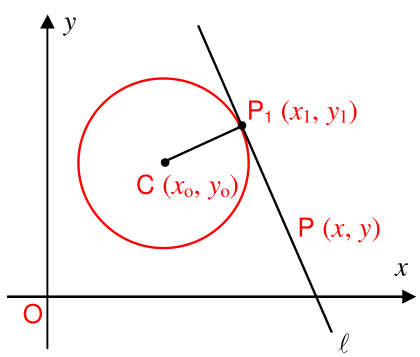


Figure 8.20

By adding $(x_1 - x_o)^2 + (y_1 - y_o)^2 = r^2$ to both sides,
we obtain

$$(x - x_1)(x_1 - x_o) + (y - y_1)(y_1 - y_o) + (x_1 - x_o)^2 + (y_1 - y_o)^2 = r^2$$

$$\Rightarrow \quad (x - x_1 + x_1 - x_o)(x_1 - x_o) + (y - y_1 + y_1 - y_o)(y_1 - y_o) = r^2$$

$$\Rightarrow (x - x_o)(x_1 - x_o) + (y - y_o)(y_1 - y_o) = r^2$$

✍Note:

If the circle is centred at the origin, then the above equation becomes:

$$x \cdot x_1 + y \cdot y_1 = r^2$$

Example 9 Find the equation of the tangent line to the circle $x^2 + y^2 = 8$ at the point $P_1(2,-2)$.

Solution The circle is centred at the origin with radius $2\sqrt{2}$. Hence the equation of the tangent line is: $2x - 2y = 8$.

Example 10 Find the equation of the tangent line to the circle $x^2 + y^2 - 4x + 6y + 4 = 0$ at $(2,0)$.

Solution By completing the square, the equation of the circle can be written as $(x - 2)^2 + (y + 3)^2 = 9$. The circle has its centre at $(2, -3)$ and radius $r = 3$. Thus, the equation of the tangent line is:

$$(x - 2)(2 - 2) + (y + 3)(0 + 3) = 9 \Rightarrow 0 + 3y + 9 = 9 \Rightarrow 3y = 0 \Rightarrow y = 0$$

The tangent line to the graph of the circle at $(2,0)$ is the horizontal line $y = 0$ (or the x-axis).

Practical application of vectors

Previously, you saw how vectors are useful in determining the equations of a line, and the equations of a tangent line to a circle. Now, you will consider practical problems and applications involving vectors.

Example 11 Show that the vectors $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (0.5, 1)$ are two parallel vectors which are of the same direction whereas the vectors $\mathbf{u}_1 = (-1, 2)$ and $\mathbf{v}_1 = (0.5, -1)$ are in opposite directions.

Solution Consider $\mathbf{u}.\mathbf{v}$ and $\mathbf{u}_1.\mathbf{v}_1$.

$$\mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \Rightarrow \frac{5}{2} = \sqrt{5} \times \frac{\sqrt{5}}{2} .\cos\theta \Rightarrow \cos\theta = 1 \text{ and hence } \theta = 0.$$

Thus, $\mathbf{u}$ and $\mathbf{v}$ are parallel and have the same direction.

$$\text{Similarly, } \mathbf{u}_1.\mathbf{v}_1 = |\mathbf{u}_1||\mathbf{v}_1|\cos\theta \Rightarrow -\frac{5}{2} = \sqrt{5} \times \frac{\sqrt{5}}{2}\cos\theta$$

$$\Rightarrow \cos\theta = -1 \text{ and hence } \theta = \pi.$$

Therefore, $\mathbf{u}_1$ and $\mathbf{v}_1$ are parallel and have opposite directions.

Example 12 If **u**, **v**, **w** and **z** are vectors from the origin to the points A, B, C and D, respectively, and **v** – **u** = **w** – **z**, prove that $ABCD$ is a parallelogram.

Solution Let O be the fixed origin of these vectors.

Since $\mathbf{v} - \mathbf{u} = \overrightarrow{AB}$ and $\mathbf{w} - \mathbf{z} = \overrightarrow{DC}$, you have $\overrightarrow{AB} = \overrightarrow{DC}$.

$\Rightarrow$The vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are parallel and equal.

Also, $\mathbf{v} - \mathbf{u} = \mathbf{w} - \mathbf{z} \Rightarrow \mathbf{w} - \mathbf{v} = \mathbf{z} - \mathbf{u} \Rightarrow \overrightarrow{BC} = \overrightarrow{AD}$

Thus, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel and equal. Hence, $ABCD$ is a parallelogram.

Example13 Prove that the sum of the three vectors determined by the medians of a triangle directed from the vertices is zero.

Solution Let ABC be a triangle and D, E, F the mid-points of the sides BC, CA, and AB, respectively, as shown in Figure 8.21.

First, consider the triangle ABD. You have

$$\overrightarrow{AD} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} \quad \ldots\ldots\ldots\ldots\ 1$$

In the same way, you see that

$$\overrightarrow{BE} = \overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA} \quad \ldots\ldots\ldots\ldots\ 2$$

and $$\overrightarrow{CF} = \overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB} \quad \ldots\ldots\ldots\ldots\ 3$$

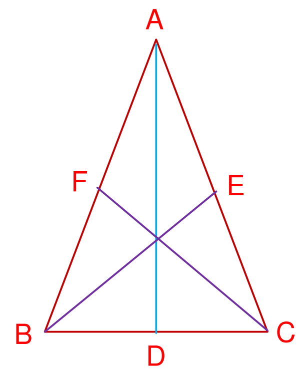


Figure 8.21

Adding up 1, 2 and 3, you get

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \frac{3}{2}(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) = \frac{3}{2}.0 = 0$$

Example14 A video camera weighing 15 pounds is going to be suspended by two wires from the ceiling of a room as shown in Figure 8.22. What is the resulting tension in each wire?

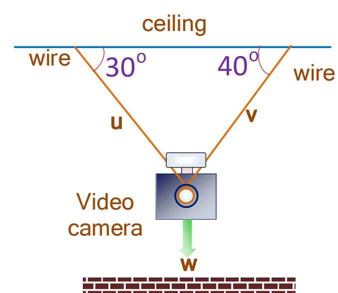


Figure 8.22

Solution The force vector of the camera is straight down, so **w** = (0, –15).

Vector **u** has magnitude $|\mathbf{u}|$ and can be represented as

$(-|\mathbf{u}|\cos 30^\circ,\ |\mathbf{u}|\sin(30^\circ))$.

Similarly, $\mathbf{v} = (|\mathbf{v}|\cos 40^\circ,\ |\mathbf{v}|\sin 40^\circ)$.

Since the system is in equilibrium, the sum of the force vectors is **0**.

$\Rightarrow \mathbf{0} = \mathbf{u} + \mathbf{v} + \mathbf{w} = (-|\mathbf{u}|\cos 30^\circ + |\mathbf{v}|\cos 40^\circ + 0,\ |\mathbf{u}|\sin 30^\circ + |\mathbf{v}|\sin 40^\circ - 15)$

From the components of the vector equation, you have two equations,

$$\begin{cases} 0 = -|\mathbf{u}|\cos 30^\circ + |\mathbf{v}|\cos 40^\circ \\ 0 = |\mathbf{u}|\sin 30^\circ + |\mathbf{v}|\sin 40^\circ - 15 \end{cases}$$

that you want to solve for the tensions $|\mathbf{u}|$ and $|\mathbf{v}|$.

From the first, you get $|\mathbf{u}|\cos 30^\circ = |\mathbf{v}|\cos 40^\circ \Rightarrow |\mathbf{v}| = |\mathbf{u}|\dfrac{\cos 30^\circ}{\cos 40^\circ}$

Substituting this value for $|\mathbf{v}|$ into the second equation you have

$$0 = |\mathbf{u}|\sin 30^\circ + |\mathbf{u}|\frac{\cos 30^\circ}{\cos 40^\circ} . \sin 40^\circ - 15$$

$$\Rightarrow |\mathbf{u}| = \frac{15}{\sin 30^\circ + (\cos 30^\circ)(\tan 40^\circ)} \cong 12.2 \text{ pounds}$$

Putting this value back into

$$|\mathbf{v}| = |\mathbf{u}|\frac{\cos 30^\circ}{\cos 40^\circ}, \text{ you get } |\mathbf{v}| = (12.2)\frac{\cos (30^\circ)}{\cos (40^\circ)} \cong 13.9 \text{ pounds.}$$

Exercise 8.5

1 Find the vector equation of the line that passes through the point P_o and is parallel to the vector **v** where

a $P_o = (-2, 1)$; $\mathbf{v} = (-1,1)$ **b** $P_o = (1,1)$; $\mathbf{v} = (2,2)$

2 Find an equation of the circle centred at C (1, –2) with radius $r = \dfrac{3}{2}$.

3 Given an equation of a line ℓ by $P = (1, 0) + t\,(2, 2), t \in \mathbb{R}$, find out whether the, points A (1, 0), B (2, 2), C(–5, –6) and D (3, 0) lie on ℓ. For those of them lying on ℓ find the respective values of the parameter t.

4 Are the points A, B and C collinear?

A A (1, –4),B (–2, –3), C (11, –11) **b** A(–2, –3), B(4, 9), C (–11, –21)

5 Find the equation (both in parametric form and standard form) of the line through the points (3, 5) and (–2, 3).

6 Show that the given point lies on the circle and find the equation of the tangent line at the point.

a $x^2 + y^2 - 2x - 4y - 9 = 0$ at $P_1(1,4)$ **b** $(x+2)^2 + y^2 = 3$ at $P_1(-1, \sqrt{2})$

7 If **u**, **v**, **w**, **z** are vectors from the origin to the points A, B, C, D, respectively, and **v** – **u** = **w** – **z**, then show that ABCD is a parallelogram.

8 Figure 8.23 shows the magnitudes and directions of six coplanar forces (forces on the same plane).

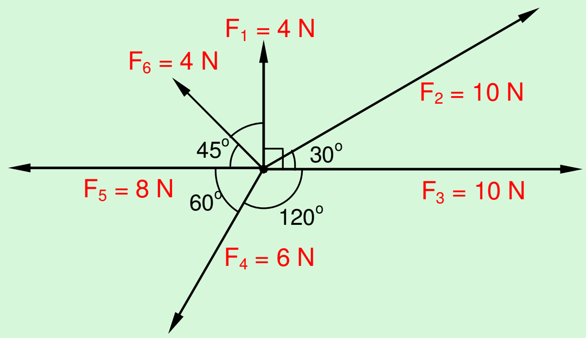


Figure 8.23

Find each of the following dot products.

a $F_1 . F_2$ **b** $F_5 . F_6$ **c** $(F_1 + F_2 – F_3) . (F_4 + F_5 – F_6)$

9 Let **a** = 3**i** + **j**, **b** = 2**i** – 2**j** and **c** = **i** + 3**j** be vectors. Find the unit vectors in the direction of each of the following vectors.

a **a** + **b** **b** 2**a** + **b** – $\frac{3}{2}$ **c**.

10 Three forces $\mathbf{F}_1 = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{F}_2 = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{F}_3 = 3\mathbf{i} - \mathbf{j}$ measured in Newton act on a particle causing it to move from **A** = **i** – 2**j** to **B** = 3**i** + 4**j** where AB is measured in meters. Find the total work done by the combined forces.

8.5 TRANSFORMATION OF THE PLANE

Transformations are of practical importance, especially in **solving problems** and **describing difficulties** in simpler forms. Transformations can be managed in different forms, those that **maintain direction** and those that **change direction**. There are many versions of transformations, but, in this section, you are going to consider three types of transformations namely **translations**, **reflections** and **rotations**.

Group work 8.4

1 When you blow up a balloon, its shape and size change.

Figure 8.24

In which of the following conditions does the shape or size or both of the object change.

a When a rubber is stretched.

b When a commercial jet flies from place to place at a specific time.

c When the earth rotates about its axis.

d When you see your image in a plane mirror.

e When you draw the map of your school compound.

2 Let T be a mapping of the plane onto itself given by T $((x, y)) = (x + 1, -y)$.

For example, T((4, 3)) = (4 + 1, – 3) = (5, –3).

If A = (0, 1), B = (–3, 2) and C = (2, 0), find the coordinates of the image ofA, Band C.

Find the image of ΔABC under T. Is ΔABC congruent to its image?

3 Suppose T is a mapping of the plane onto itself which sends point P to point P'

Let A = (2, –3) and B = (5, 4). Compare the lengths of AB and A'B' when

a T $((x, y)) = (x, 0)$ **b** T $((x, y)) = (x, -y)$

c T $((x, y)) = (x + 1, y - 3)$ **d** T $((x, y)) = \left(\frac{1}{2}x, 2y\right)$

4 Can you list some other transformations?

In this Group Work you saw that some mappings called **Transformations** of the plane onto itself preserve shape, size or distance between any two points. Based on this, transformations are classified as either rigid motion or non rigid motion.

Definition 8.10 Rigid motion

A motion is said to be **rigid motion**, if it preserves distance. That is for P ≠ Q, PQ = P'Q' where P' and Q' are the images of P and Q, respectively. Otherwise it is said to be non-rigid motion.

A transformation is said to be an **identity transformation**, if the image of every point is itself. For example, if an object is rotated 360^o it is an identity transformation.

Note:

✓ Rigid motion carries any plane figure to a congruent plane figure, i.e., it carries triangles to congruent triangles, rectangles to congruent rectangles, etc.

An **identity transformation** is a rigid motion.

In this topic three different types of rigid motions are presented.

Translations | **Reflections** | **Rotations**

Figure 8.25

8.5.1 Translation

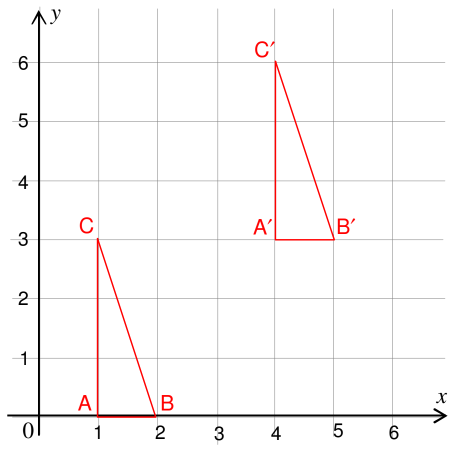


Figure 8.24

When ΔABC is transformed to $\Delta A'B'C'$, AB and $A'B'$ are parallel to the x-axis, and AC and $A'C'$ are parallel to the y-axis. Moreover, ΔABC and $\Delta A'B'C'$ have the same orientation. i.e., the way they face is the same. This type of transformation is said to be a **translation**.

> **Definition 8.11**
>
> If every point of a figure is moved along the same direction through the same distance, then the transformation is called a **translation** or **parallel movement**.

If point P is translated to point P', then the vector $\overrightarrow{PP'}$ is said to be the **translation vector**.

If $\mathbf{u} = (h, k)$ is a translation vector, then the image of the point P(x, y) under the translation will be the point P′$(x + h, y + k)$.

Example1 Let T be a translation that takes the origin to (1, 2). Determine the translation vector and find the images of the following points.

a (2,–1) **b** (–3, 5) **c** (1, 2)

Solution T ((0, 0)) = (1, 2) $\Rightarrow$ **u** = (1, 2) is the translation vector.

$\Rightarrow x \mapsto x + 1$ and $y \mapsto y + 2$

Thus,

a T ((2, –1)) = (2 + 1, –1 + 2) = (3, 1)

b T ((–3, 5)) = (–3 + 1, 5 + 2) = (–2, 7)

c T ((1, 2)) = (1 + 1, 2 + 2) = (2, 4).

Example2 Let the points P (x_1, y_1) and Q (x_2, y_2) be translated by the vector **u**= (h, k). Show that $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{P'Q'}\right|$.

Solution Clearly P'= $(x_1 + h, y_1 + k)$ and Q′ = $(x_2 + h, y_2 + k)$.

Then, $\left|\overrightarrow{P'Q'}\right| = \sqrt{(x_2 + h - x_1 - h)^2 + (y_2 + k - y_1 - k)^2}$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \overrightarrow{PQ}.$$

The above example shows that a translation is a rigid motion. You can state a translation formula in terms of coordinates as follows:

1 If (h, k) is a the translation vector, then

a the origin is translated to (h, k) i.e., $(0, 0) \rightarrow (h, k)$

b the point $P(x, y)$ is translated to $P'(x + h, y + k)$.

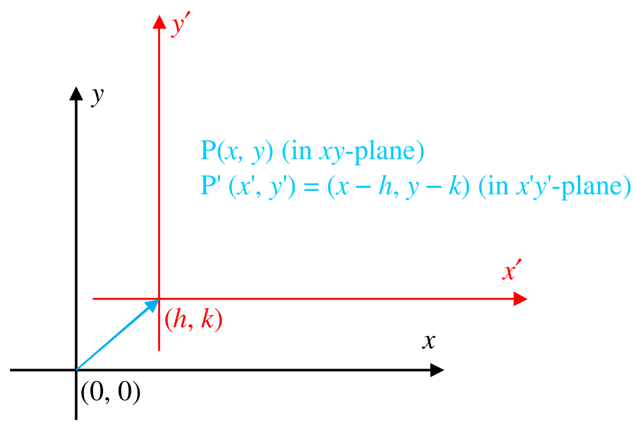


Figure 8.25

2 If the translation vector is $\overline{AB}$ where A = (a, b) and B = (c, d), then

a the origin is translated to $(c – a, d – b)$, and

b the point $P(x, y)$ is translated to $(x + c – a, y + d – b)$

Example3 If a translation T takes the origin to P'(1, 2), then

T $(x, y) = (x + 1, y + 2)$ and T (–2, 3) = (–2 + 1, 3 + 2) = (–1, 5).

Example 4 If a translation T takes the origin to (–1, 1), then find

a the images of the points P (1, 3) and Q (–3, 6)

b the image of the triangle with vertices A(2, –2), B (–3, 2) and C(4, 1)

c the equation of the image for the circle whose equation is $x^2 + y^2 = 4$.

Solution

a The image of the point P (1, 3) is T (1, 3) = (1 + (–1), 3 + 1) = (0, 4).

The image of the point Q (–3, 6) is T (–3, 6) = (–3 – 1, 6 +1) = (–4, 7)

b T (2, –2) = (2 + (–1), –2 + 1) = (1,–1)

T (–3, 2) = (–3 + (–1), 2 + 1) = (–4, 3)

T (4, 1) = (4 + (–1), 1 + 1) = (3, 2)

Thus, A' = (1, –1), B' = (–4, 3) and C' = (3, 2)

The image of Δ ABC is ΔA'B'C'.

c The image of (x, y) under T is T $(x, y) = (x – 1, y + 1)$.

The centre of the circle (0, 0) is translated to (–1, 1)

Thus, the image of $x^2 + y^2 = 4$ is $(x+ 1)^2 + (y– 1)^2 = 4$

Example 5 If a translation T takes the point (–1, 3) to the point (4, 2), then find the images of the following lines under the translation T.

a $\ell : y = 2x – 3$ **b** $\ell : 5y + x = 1$

Solution The translation vector is $(h, k) = (4 – (–1), 2 – 3) = (5, –1)$. Thus, the point P$(x, y)$ is translated to the point P'$(x + 5, y – 1)$.A translation maps lines onto parallel lines. Let ℓ' be the image of ℓ under T. Then,

a $\ell' : y – (–1) = 2 (x– 5) – 3$

$\Rightarrow \ell : y = 2x – 14$

b $\ell' : 5(y + 1) + (x – 5) =1$

$\Rightarrow \ell' : 5y + x = 1 \Rightarrow \ell ' = \ell$. *Explain!*

Example 6 Determine the equation of the curve $2x^2 + 3y^2 - 8x + 6y = 7$ when the origin is translated to the point A(2, –1).

Solution The translation vector is $(h, k) = (2, –1)$. Thus, the point P(x, y) is translated to the point P'$(x + 2, y – 1)$.Substituting $x – 2$ and $y + 1$ in the equation, you obtain $2(x-2)^2 + 3(y+1)^2 - 8(x-2) + 6(y+1) = 7$.

Expanding and simplifying, the equation of the curve becomes

$$2x^2 + 3y^2 - 16x + 12y + 26 = 0$$

Exercise 8.6

1. If a translation T takes the origin to the point A(–3, 2), find the image of the rectangle ABCD with vertices A(3, 1), B(5, 1), C(5, 4) and D(3, 4).
2. Triangle ABC is transformed into triangle A'B'C' by the translation vector (4, 3). If A = (2, 1), B = (3, 5) and C = (–1, –2), find the coordinates of A', B' and C'.
3. Quadrilateral ABCD is transformed into A'B'C'D' by a translation vector (3, –2).

   If A = (1, 2), B = (3, 4), C = (7, 4) and D = (2, 5), then find A', B', C' and D' and draw the quadrilaterals ABCD and A'B'C'D' on graph paper.
4. What is the image of a circle under a translation?
5. Find the equation of the image of the circle $(x + 1)^2 + (y - 3)^2 = 5$ when translated by the vector $\overrightarrow{PQ}$, where P = (1, –1) and Q = (–4, 3).
6. A translation T takes the origin to A(3, –2). A second translation S takes the origin to B(–2, –1). Find where T followed by S takes the origin, and where S followed by T takes the origin.
7. If a translation T takes (2, –5) to (–2, 1), find the image of the line $\ell: 2x - 3y = 7$.
8. If a translation T takes the origin to (4, –5), find the image of each of the following lines.

   a $y = 3x + 7$ **b** $4y + 5x = 10$
9. If the point A(3, –2) is translated to the point A'(7, 10), then find the equation of the image of

   a the ellipse $4x^2 + 3y^2 - 2x + 6y = 0$ **b** the parabola $y^2 = 4x$

   c the hyperbola $xy = 1$ **d** the function $f(x) = x^3 - 3x^2 + 4$

8.5.2 Reflections

As the name indicates, reflection transforms an object using a reflecting material.

ACTIVITY 8.4

1. Using the concept "reflection by a plane mirror", find the images of the following figures by considering line L as a mirror line.

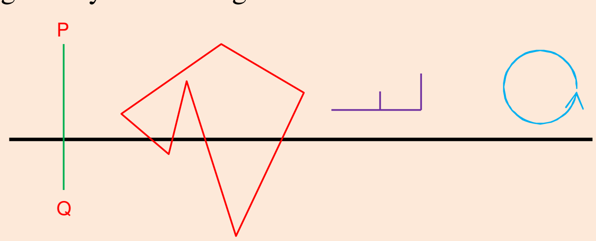


Figure 8.26

2 In Figure 8.27a below A' is the mirror image of A. Copy the figure and draw the reflecting line.

3 In Figure 8.27b below A' and B' are the images of A and B, respectively. Copy the figure and determine the reflection line.

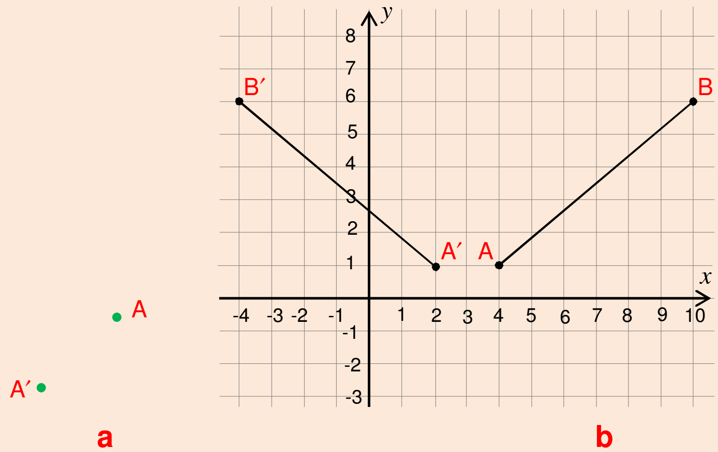


Figure 8.27

4 Discuss the conditions that are necessary to define reflection.

Definition 8.12

Let L be a fixed line in the plane. A reflection M about a line L is a transformation of the plane onto itself which carries each point P of the plane into the point P' of the plane such that L is the **perpendicular bisector** of PP'.

The line L is said to be the line of reflection or the axis of reflection.

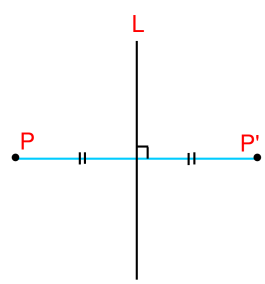


Figure 8.28

✍Note:

Every point on the axis of reflection is its own image.

NOTATION:

The reflection of point P about the line L, is denoted by M (P), i.e. P'=M(P).

Reflection has the following properties:

1. A reflection about a line L has the property that, if for two points P and Q in the plane, P = Q, then M(P) = M(Q). Hence, reflection is a function from the set of points in the plane into the set of points in the plane.
2. A reflection about a line L maps distinct points to distinct points, i.e., if P≠Q, then M(P) ≠ M(Q). Equivalently, it has the property that if, for two points P, Q in the plane, M(P) = M(Q), then P = Q. Thus, reflection is a one-to-one mapping.
3. For every point P' in the plane, there exists a point P such that M(P) = P'. If the point P' is on L, then there exists P = P' such that M(P) = P'. Thus, reflection is an onto mapping.

Theorem 8.5

A reflection *M* is a rigid motion. That is, if P' = M(P) and Q' = M(Q), then PQ =P'Q'.

We now consider reflections with respect to the axes and the lines $y = mx+b$.

A Reflection in the x and y-axes

ACTIVITY 8.5

1. Find the image of $f(x) = e^x$, when it is reflected
   a. in the y-axis b. in the x-axis c. in the line $y = x$
2. Discuss how to determine the images of points P(x, y), lines ℓ; $y = mx + b$ and circles $(x-h)^2 + (y-k)^2 = r^2$, when they are reflected in each of the following lines
   a. $y = 0$ (x-axis) b. $x = 0$ (y-axis) c. $y = x$ d. $y = -x$

B Reflection in the line $y = mx$, where $m = \tan\theta$

Let ℓ be a line passing through the origin and making an angle θ with the positive x-axis.

Then the slope of ℓ is given by $m = \tan\theta$ and its equation is $y = mx$. *See* Figure 8.29.

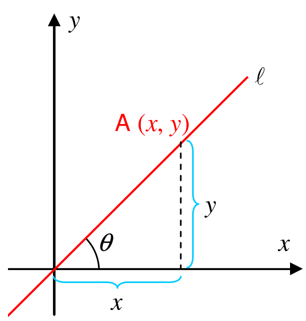


Figure 8.29

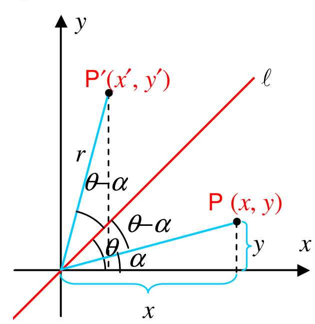


Figure 8.30

You will now find the image of a point P(x, y) when it is reflected about this line.

See Figure 8.30

Let P'(x', y') be the image of P(x, y).

The coordinates of P are:

$$x = r\cos\alpha \text{ and } y = r\sin\alpha$$

The coordinates of P' are:

$$x' = r\cos(2\theta - \alpha) \text{ and } y' = r\sin(2\theta - \alpha)$$

Expanding cos ($2\theta - \alpha$) and sin ($2\theta - \alpha$),

Now, use the following trigonometric identities that you will learn in Section 9.4.2.

1 Sine of the sum and the difference

- ✓ $\sin(x + y) = \sin x\cos y + \cos x\sin y$
- ✓ $\sin(x - y) = \sin x\cos y - \cos x\sin y$

2 Cosine of the sum anddifference

- ✓ $\cos(x + y) = \cos x\cos y - \sin x\sin y$
- ✓ $\cos(x - y) = \cos x\cos y + \sin x\sin y$

Using these trigonometric identities, you obtain:

$$x' = r[\cos 2\theta\cos\alpha + \sin 2\theta\sin\alpha] = (r\cos\alpha)\cos 2\theta + (r\sin\alpha)\sin 2\theta,$$

$$= x\cos 2\theta + y\sin 2\theta, \text{ and}$$

$$y' = r[\sin 2\theta\cos\alpha - \sin\alpha\cos 2\theta] = (r\cos\alpha)\sin 2\theta - (r\sin\alpha)\cos 2\theta,$$

$$= x\sin 2\theta - y\cos 2\theta$$

Thus, the coordinates of P'(x', y'), the image of the point P (x, y) when reflected about the line $y = mx$ is:

$$x' = x\cos 2\theta + y\sin 2\theta$$

$$y' = x\sin 2\theta - y\cos 2\theta$$

where θ is the angle of inclination of the line $\ell: y = mx$

Based on the value of θ, you will have the following four special cases:

1 When $\theta = 0$, you will have reflection in the x-axis. Thus, (x, y) is mapped to(x, $-y$)

2 When $\theta = \frac{\pi}{4}$, you will have reflection about the line $y = x$ and hence (x, y) is mapped to (y, x).

3 When $\theta = \frac{\pi}{2}$, you will have reflection in the y – axis and (x, y) is mapped to $(-x, y)$.

4 When $\theta = \frac{3\pi}{4}$, you will have reflection about the line $y = -x$ and (x, y) is mapped to $(-y, -x)$.

Example 7 Find the images of the points (3, 2), (0, 1) and (–5, 7) when reflected about the line $y = mx$, where $m = \tan\theta$ and $\theta = \frac{\pi}{4}$

Solution: This is actually a reflection about the line $y = x$. Thus, the images of (3, 2), (0, 1) and (–5, 7) are (2, 3), (1, 0) and (7, –5), respectively.

Example 8 Find the images of the points P (3, 2), Q (0, 1) and R (–5, 7) when reflected about the line $y = \frac{1}{\sqrt{3}}x$.

Solution Since $\tan\theta = \frac{1}{\sqrt{3}}$, you have $\theta = \frac{\pi}{6}$. Thus, if $P'(x', y')$ is the image of P, then

$$x' = x\cos 2\theta + y\sin 2\theta = 3\cos\left(\frac{\pi}{3}\right) + 2\sin\left(\frac{\pi}{3}\right) = 3\times\frac{1}{2} + 2\times\frac{\sqrt{3}}{2} = \frac{3+2\sqrt{3}}{2}$$

$$y' = x\sin 2\theta - y\cos 2\theta = 3\sin\left(\frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right) - 2\times\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{2} - 1$$

Hence, the image of P (3, 2) is $P'\left(\frac{3+2\sqrt{3}}{2}, \frac{3\sqrt{3}}{2} - 1\right)$

Similarly, you can show that the images of Q (0, 1) and R(–5, 7) are $Q'\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $R'\left(\frac{-5+7\sqrt{3}}{2}, \frac{-5\sqrt{3}-7}{2}\right)$, respectively.

Example9 Find the image of A = (1,–2) after it has been reflected in the line $y = 2x$.

Solution $y = 2x \Rightarrow y = (\tan\theta)x \Rightarrow \theta = \tan^{-1}(2)$.

But, from trigonometry, you have

$$\sin\theta = \frac{2}{\sqrt{5}} \text{ and } \cos\theta = \frac{1}{\sqrt{5}} \Rightarrow \cos(2\theta) = \cos^2\theta - \sin^2\theta = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5},$$

$$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{4}{5} \Rightarrow x' = -\frac{3}{5}x + \frac{4}{5}y \text{ and } y' = \frac{4}{5}x + \frac{3}{5}y$$

$$\Rightarrow M((1, -2)) = \left(-\frac{11}{5}, -\frac{2}{5}\right)$$

✍Note:

1 If a line ℓ' is perpendicular to the axis of reflection L, then L' is its own image.

2 If the centre of a circle C is on the line of reflection L, then the image of C is itself.

3 If the centre O of a circle C has image O' when reflected about a line L, then the image circle has centre O' and radius the same as C.

4 If ℓ' is a line parallel to the line of reflection L, to find the image of L' when reflected about L, we follow the following steps.

Step a: Choose any point P on ℓ'

Step b: Find the image of P, M(P) = P'

Step c: Find the equation of ℓ', which is the line passing through P' with slope equal to the slope of ℓ.

C Reflection in the line $y = mx + b$

Let $\ell : y = mx + b$ be the line of reflection, where $m \in \mathbb{R} \setminus \{0\}$.

Let P(x, y) be a point in the plane, not on ℓ.

Let P'(x', y') be the image of P(x, y) when reflected about the line ℓ.

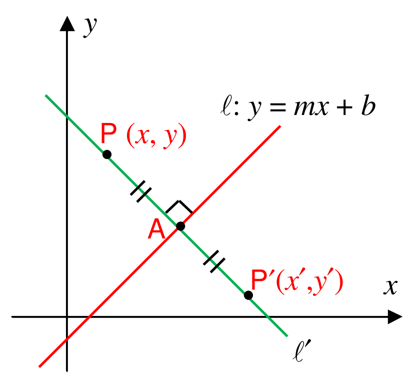


Figure 8.31

Let ℓ' be the line passing through the points P (x, y) and P' (x', y').Then, ℓ' is perpendicular to ℓ, since ℓ is perpendicular to $\overline{PP'}$.Since the slope of ℓ is m, the slope of ℓ' is $-\frac{1}{m}$. Thus, one can determine the equation of the line ℓ'. If A is the point of intersection of ℓ and ℓ', taking A as the mid-point of $\overline{PP'}$, we can find the coordinates of P'.

Thus, to find the image of a point P(x, y) when reflected about a line ℓ, we follow the following four steps.

Step 1: Find the slope of the line ℓ, say m.

Step 2: Find the equation of the line ℓ', which passes through the point P(x, y) and has slope $-\frac{1}{m}$

Step 3: Find the point of intersection A of ℓ and ℓ' which serves as the midpoint of $\overline{PP'}$.

Step 4: Using A as the mid-point of $\overline{PP'}$, find the coordinates of P'.

Example 10 Find images of the following lines and circles after reflection in the line $y = 2x - 3$.

a $2y + x = 1$ **b** $y = 2x + 1$ **c** $y = 3x + 4$

d $x^2 + y^2 - 4x - 2y + 4 = 0$ **e** $x^2 + y^2 - 2x + 3y = 8$

Solution

a The image of ℓ: $2y + x = 1$ is itself. Explain!

b ℓ: $y = 2x + 1$ is parallel to the reflecting axis.

Hence $\ell' : y = 2x + b$. We need to determine b.

Let (a, b) be any point on ℓ, say (0, 1), so that its image lies on ℓ'.

By the above reflecting procedure,

$$M((0, 1)) = (a', b') \Rightarrow \frac{b'-1}{a'-0} = -\frac{1}{2} \Rightarrow a' = -2b' + 2$$

Also, the midpoint of (0, 1) and (a', b') which is $\left(\frac{a'}{2}, \frac{b'+1}{2}\right)$ lies on the reflecting axis $\Rightarrow \frac{b'+1}{2} = 2\left(\frac{a'}{2}\right) - 3 \Rightarrow a' = \frac{b'}{2} + \frac{7}{2}$,

But $a' = -2b' + 2 \Rightarrow 2b' + 2 = \frac{b'}{2} + \frac{7}{2}$

$$\Rightarrow b' = -\frac{3}{5} \Rightarrow a' = \frac{16}{5} \quad \Rightarrow \left(\frac{16}{5}, -\frac{3}{5}\right) \text{ lies on } \ell'$$

$$\Rightarrow -\frac{3}{5} = 2\left(\frac{16}{5}\right) + b \quad b = -7 \qquad \Rightarrow \ell' : y = 2x - 7$$

c ℓ: $y = 3x + 4$ and the axis of reflection $y = 2x - 3$ meet at (-7, -17)

Next, take a point on ℓ say (0, 4) and find its image (a', b') so that ℓ passes through (a', b'). Perform the technique similar to the problem in b

Thus, $\frac{b'-4}{a'-0}=-\frac{1}{2}$ and $\frac{4+b'}{2}=2\left(\frac{a'}{2}\right)-3 \Rightarrow a'=\frac{28}{5}$ and $b'=\frac{6}{5}$

$\Rightarrow \ell'$: y= ℓ': $y=\frac{91}{63}x-\frac{434}{63}$

d $x^2 + y^2 - 4x - 2y + 4 = 0 \Rightarrow (x-2)^2 + (y-1)^2 = 1$

This is a circle of radius 1 unit with centre (2, 1) that is on $y = 2x - 3$.

$\Rightarrow$ The centre of the circle lies on the axis of reflection. Therefore, the circle is its own image.

e $x^2 + y^2 - 2x + 3y = 8 \Rightarrow (x-1)^2 + (y+\frac{3}{2})^2 = \frac{45}{4}$

The centre $(1,-\frac{3}{2})$ has image $\left(\frac{3}{5},-\frac{13}{10}\right)$

$\Rightarrow$ The image circle is $\left(x-\frac{3}{5}\right)^2+\left(y+\frac{13}{10}\right)^2=\frac{45}{4}$

Example 11 Find the image of (–1, 5) when reflected about the lines

a $y=-1$ **b** $x = 1$ **c** $y = x + 2$ **d** $y = 2x + 5$

Solution

a The image of the point (–1, 5) when reflected about the line $y=-1$ is $(-1,-7)$

b The image of the point (–1, 5) when reflected about the line $x = 1$ is (3, 5)

c The slope of $y = x + 2$ is 1.

Let P'(x', y') be the image of P (–1, 5). If ℓ' is the line passing through P and P', then its slope is $\frac{-1}{1}=-1$. Thus, the equation of ℓ' is:

$$\frac{y-5}{x+1}=-1 \Rightarrow \ell': y = -x + 4$$

The point of intersection of ℓ and ℓ' is (1, 3). Taking (1, 3) as a midpoint of $\overline{PP'}$, we get,

$$\frac{-1+x'}{2}=1 \text{ and } \frac{5+y'}{2}=3 \Rightarrow -1+x'=2 \text{ and } 5+y'=6$$

$$\Rightarrow x'=3 \text{ and } y'=1$$

Therefore, the image of P (–1, 5) is P' (3, 1).

d The slope of $y = 2x + 5$ is 2. If P'(x', y') is the image of P(–1, 5) and ℓ' is the line through P and P', then its slope is $\frac{-1}{2}$. Thus, the equation of ℓ' is:

$$\frac{y-5}{x+1}=\frac{-1}{2} \Rightarrow \ell' : y = \frac{-1}{2}x+\frac{9}{2}$$

The point of intersection of ℓ and ℓ' is A$\left(\frac{-1}{5},\frac{23}{5}\right)$. Taking A as the midpointof $\overline{PP'}$, find the coordinates of P' as:

$$\frac{-1+x'}{2}=\frac{-1}{5} \text{ and } \frac{5+y'}{2}=\frac{23}{5} \Rightarrow -5 + 5x' = -2 \text{ and } 25 + 5y' = 46$$

$$\Rightarrow 5x' = 3 \text{ and } 5y' = 46 - 25 = 21 \Rightarrow x' = \frac{3}{5} \text{ and } y' = \frac{21}{5}$$

Hence, the image of P (–1, 5) is P'$\left(\frac{3}{5}, \frac{21}{5}\right)$.

Example12 Given the equation of the circle $x^2+(y-1)^2 = 1$, find the equation of its graph after a reflection about the line $y = x$.

Solution The centre of the circle is (0, 1). The reflection of (0, 1) about the line $y = x$ is (1, 0), which is the centre of the image circle. Therefore, the equation of the image circle is $(x-1)^2 + y^2 = 1$

Example 13 Find the image of the line ℓ': $y = -3x - 7$ after a reflection about the line ℓ: $y = -3x + 1$

Solution Pick a point P on ℓ', say P (1, –10).

To find the image of the point P (1, –10) when reflected about the line$y=-3x+1$, proceed as follows:

Since slope of ℓ is –3, the slope of the perpendicular line is $\frac{1}{3}$. Thus, the equation of the line through (1, –10) with slope $\frac{1}{3}$ is: $\frac{y+10}{x-1}=\frac{1}{3}$

$$\Rightarrow y = \frac{1}{3}x - \frac{31}{3}$$

The point of intersection of $y=-3x+1$ and $y=\frac{1}{3}x-\frac{31}{3}$ is $A\left(\frac{34}{10},\frac{-92}{10}\right)$.

Taking A as a mid-point of $\overline{PP'}$, find the coordinates of the image $P'(x', y')$ of P, i.e.,

$$\frac{1+x'}{2}=\frac{34}{10} \text{ and } \frac{-10+y'}{2}=\frac{-92}{10}$$

$$\Rightarrow 10+10x'=68 \text{ and } -100+10y'=-184$$

$$\Rightarrow x'=\frac{58}{10} \qquad \text{and} \qquad y'=\frac{-84}{10}$$

Therefore, the image of $P(1,-10)$ is $P'\left(\frac{58}{10},\frac{-84}{10}\right)$.

Now, you need to find the equation the line passing through P' with slope –3, i.e.,

$$\frac{y+\frac{84}{10}}{x-\frac{58}{10}}=-3 \Rightarrow \frac{10y+84}{10x-58}=-3$$

$$\Rightarrow 10y+84=-30x+174$$

$$\Rightarrow 10y=-30x+174-84$$

$$\Rightarrow 10y=-30x+90$$

$$\Rightarrow y=-3x+9$$

Hence, the image of the line $\ell': y=-3x-7$ when reflected about the line

$y=-3x+1$ is $y=-3x+9$

Example14 Find the image of the circle $(x-1)^2+(y+5)^2=1$, when it is reflected about the line $y=2x-1$.

Solution The centre of the circle is $(1,-5)$, the image of the point $(1,-5)$ when reflected about the line $\ell: y=2x-1$ is $\left(\frac{-19}{5},\frac{-13}{5}\right)$

Thus, the equation of the image circle is $\left(x+\frac{19}{5}\right)^2+\left(y+\frac{13}{5}\right)^2=1$

Exercise 8.7

1 The vertices of triangle ABC are A (2, 1), B (3, –2) and C (5, –3). Give the coordinates of the vertices after:

   a a reflection in the x-axis
   b a reflection in the y-axis
   c a reflection in the line $x + y = 0$
   d a reflection in the line $y = x$.

2 Find the image of the point (–4, 3) after a reflection about the line $\ell: y = x - 2$

3 If the image of the point (–1, 2) under reflection is (1, 0), find the line of reflection.

4 Find out some of the figures which are their own images in reflection about the line $y = x$.

5 Find the image of the line $\ell: y = x + 4$ after it has been reflected about the line $L: y = x - 3$

6 Find the image of the line $\ell: y = 2x + 1$ after it has been reflected about the line $L: y = 3x + 2$

7 Given an equation of a circle $(x-2)^2 + (y-3)^2 = 25$, find the equation of the image circle after a reflection about the line $y = x + 3$.

8 The image of the circle $x^2 + y^2 - x + 2y = 0$ when it is reflected about the line L is $x^2 + y^2 - 2x + y = 0$. Find the equation of L.

9 If T is a translation that sends (0, 0) to (3, –2) and M is a reflection that maps (0, 0) to (2, 4), find

   a T(M(1, 3))
   b M(T(1, 3))

10 In a reflection, the image of the line $y - 2x = 3$ is the line $2y - x = 9$. Find the axis of reflection.

8.5.3 Rotations

Rotation is a type of transformation in which figures turn around a point called the centre of rotation. The following Group Work will introduce you the idea of rotation.

Group work 8.5

1 In the following figure, A, B, C and D are points on the same circle with centre at the origin. The chords $\overline{AC}$ and $\overline{BD}$ are perpendicular.

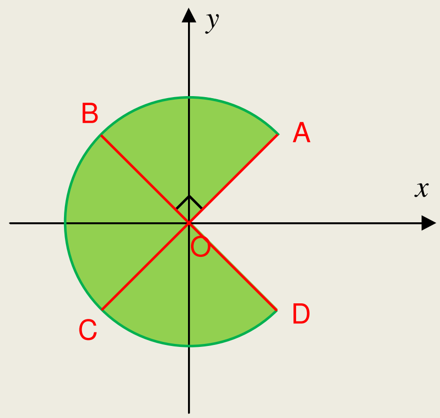


Figure 8.32

Discuss the following questions in groups.

a If A = (2, 3) find the coordinates of B, C and D.

b If A = (x , y) express the coordinates of B, C and D in terms of x and y.

2 Look at the figure below.

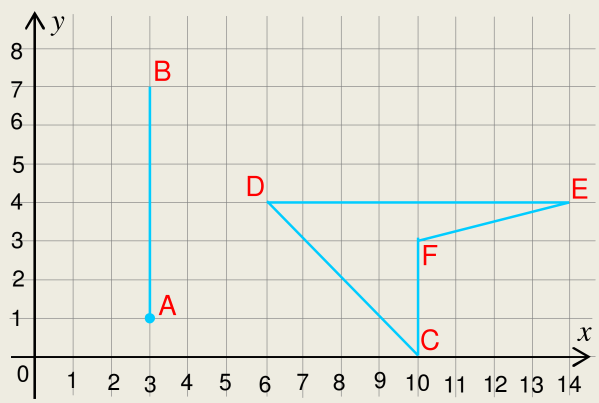


Figure 8.33

By placing a piece of transparent paper on this figure, trace $\overline{AB}$ and $CDEF$.

Hold a pencil at the origin and rotate the paper 90^{o}counter clockwise. After this rotation, write the images of A, B, C, D, E and F to be A', B', C', D', E' and F', respectively, on the paper.

a Find the coordinates of those points on the transparent paper by referring the x and y coordinates of the original figure.

b Is there a fixed point in this rotation?

c Discuss whether or not this transformation is a rigid motion.

d What do you think the images of the x and y axes are?

3 Discuss what you need to define rotation.

In the Group work, you have seen a third type of transformation called **rotation**. Rotation is formally defined as follows.

Definition 8.13

A rotation R about a point O through an angle θ is a transformation of the plane onto itself which carries every point P of the plane into the point P' of the plane such that $OP = OP'$ and m ($\angle POP'$) = θ. O is called the **centre of rotation** and θ is called the **angle of rotation**.

Note:

i The rotation is in the **counter clockwise** direction, if $\theta > 0$ and in the **clockwise** direction if $\theta < 0$.

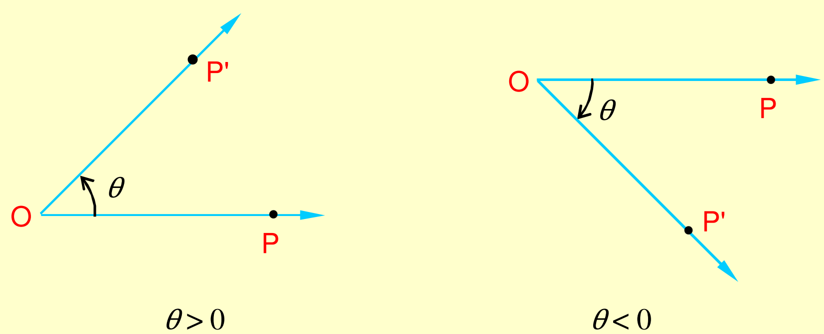


Figure 8.34

ii Rotation is a rigid motion.

Example 15 Find the image of point A (1, 0) when it is rotated through 30° about the origin.

Solution Let the image of A (1, 0) be A' (a, b) as shown in the figure.

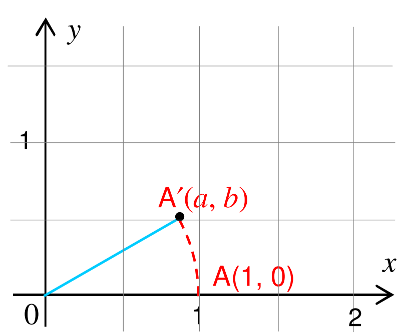


Figure 8.35

But from trigonometry, $(a, b) = (r \cos \theta, r \sin \theta)$ where $r = 1$ and $\theta = 30^{o}$ in this example. Therefore, the image of A (1, 0) is A' $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

NOTATION:

If R is rotation through an angle θ, then the image of P (x, y) is denoted by $R_{\theta}(x, y)$. In the above example, $R_{30^{o}}(1, 0) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

At this level, we derive a formula for a rotation R about O (0, 0) through an angle θ.

Theorem 8.6

Let R be a rotation through angle θ about the origin. If $R_{\theta}(x, y) = (x', y')$,

then $x' = x\cos\theta - y \sin \theta$

$y' = x \sin \theta + y\cos\theta$

Proof

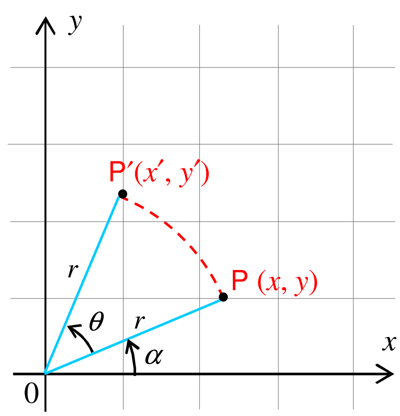


Figure 8.36

From trigonometry we have,

$(x, y) = (r \cos \alpha, r \sin \alpha)$ and $(x', y') = (r \cos (\alpha + \theta), r \sin (\alpha + \theta))$

$\Rightarrow r \cos (\alpha + \theta) = r \cos \alpha \cos \theta - r \sin \alpha \sin \theta$

$= x \cos \theta - y \sin \theta$

$r \sin (\alpha + \theta) = r \sin \alpha \cos \theta + r \cos \alpha \sin \theta$

$= y \cos \theta + x \sin \theta$

$\therefore R_{\theta}(x, y) = (x \cos \theta - y \sin \theta, y \cos \theta + x \sin \theta)$

✍Note:

Let R be a counter-clockwise rotation through an angle θ about the origin. Then

i $\theta = \frac{\pi}{2} \Rightarrow R(x, y) = (-y, x)$

ii $\theta = \pi \Rightarrow R(x, y) = (-x, -y)$

iii $\theta = \frac{3\pi}{2} \Rightarrow R(x, y) = (y, -x)$

iv $\theta = 2n\pi$ for $n \in \mathbb{Z} \Rightarrow$ R is the identity transformation.

v Every circle with centre at the centre of rotation is fixed.

Example 16 Using the formula, find the images of the following points in rotation about the origin through the indicated angle.

a (4, 0) ; 60^{o} **b** (1, 1) ; $-\frac{\pi}{6}$ **c** (1, 2); 450^{o}

Solution

a

$x' = x\cos\theta - y\sin\theta;\ x = 4,\ y = 0; \theta = 60^{o}$

$= 4\cos 60^{o} - 0 \times \sin 60^{o} = 2$

$y' = x\sin\theta + y\cos\theta$

$= 4\sin 60^{\circ} + 0 \times \cos 60^{\circ} = 2\sqrt{3}$

$\Rightarrow R_{60^{o}}(4, 0) = \left(2, 2\sqrt{3}\right)$

b

$x' = 1 \times \cos\left(-\frac{\pi}{6}\right) - 1 \times \sin\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}$

$y' = x\sin\theta + y\cos\theta$

$y' = 1 \times \sin\left(\frac{-\pi}{6}\right) + 1 \times \cos\left(\frac{-\pi}{6}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}$

$\Rightarrow R_{60^{o}}(1, 1) = \left(\frac{\sqrt{3}}{2} + \frac{1}{2},\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$

c

$x' = 1 \times \cos\left(\frac{\pi}{2}\right) - 2 \times \sin\left(\frac{\pi}{2}\right)$

$x' = 1 \times \cos\left(\frac{\pi}{2}\right) - 2 \times \sin\left(\frac{\pi}{2}\right)$

$x' = -2(1) = -2$

$y' = 1 \times \sin\left(\frac{\pi}{2}\right) + 2 \times \cos\left(\frac{\pi}{2}\right)$

$y' = 1 \times 1 + 2 \times 0$

$\Rightarrow y' = 1$

Notice that $450^o = (360^o + 90^o)$

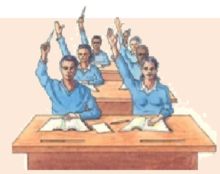
$\therefore R\,(x,\, y) = (-y,\, x)$

$\therefore R\,(1,\, 2) = (-2,\, 1)$

Rotation when the centre of rotation is (x_o, y_o)

So far you have seen rotation about the origin. The next activity introduces rotation about an arbitrary point (x_o,y_o).

ACTIVITY 8.6



1 In the following figure,a rotation R sends A to A' and B to B'.

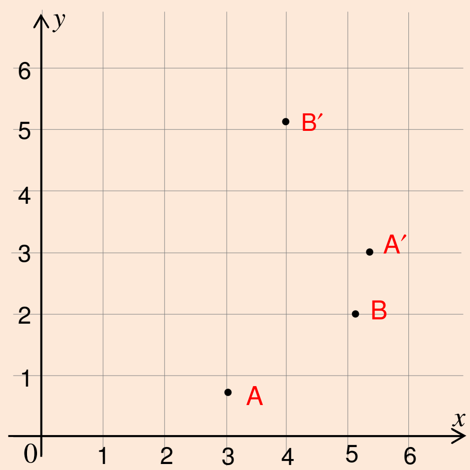


Figure 8.37

Discuss how to determine the centre of rotation.

2 If R is a rotation through $\frac{\pi}{4}$ about A(3, 2), discuss how to determine the image of a point P (2, 0).

The above activity leads to the following generalized formula.

Corollary 8.4

If P' (x', y') is the image of P (x, y), after it has been rotated through an angle θ about (x_o, y_o), then

$$x' = x_o + (x - x_o)\cos\theta - (y - y_o)\sin\theta$$
$$y' = y_o + (x - x_o)\sin\theta + (y - y_o)\cos\theta$$

Note:

As in the case of translation and reflection, to find the image of a circle under a given rotation we follow the following steps:

1. Find the centre and radius of the given circle
2. Find the image of the centre of the circle under the given rotation.
3. Equation of the image circle will be an equation of the circle centred at the image of the centre of the given circle with radius the same as the radius of the given circle.

Example 17 Find the image of the circle $(x-3)^2+(y+5)^2=1$ when it is rotated through $\frac{5\pi}{3}$ about (4, –3).

Solution According to the note given above, we compute only the image of the centre of the circle. The centre is (3, –5) and its radius is 1 unit.

$$x' = x_o + (x-x_o)\cos\theta - (y-y_o)\sin\theta$$

Where $(x, y) = (3, -5)$; $(x_o, y_o) = (4, -3)$; $\theta = \frac{5\pi}{3}$

$$x' = 4+(3-4)\cos\frac{5\pi}{3} - (-5+3)\sin\frac{5\pi}{3} = 4-\frac{1}{2}+2\left(-\frac{\sqrt{3}}{2}\right) = \frac{7}{2}-\sqrt{3}$$

$$y' = y_o + (x-x_o)\sin\theta + (y-y_o)\cos\theta$$

$$\Rightarrow y' = -3+(3-4)\sin\frac{5\pi}{3}+(-5-(-3))\cos\frac{5\pi}{3} = -3+\frac{\sqrt{3}}{2}-1 = -4+\frac{\sqrt{3}}{2}$$

Thus, the equation of the image of the circle is $\left(x-\frac{7}{2}+\sqrt{3}\right)^2+\left(y+4-\frac{\sqrt{3}}{2}\right)^2=1$

Note:

One can also obtain the image of a line under a given rotation as follows:

✓ Choose two points on the line.

✓ Find the images of the two points under the given rotation.

Thus, the image line will be the line passing through the two image points.

Example18 Find the equation of the line $\ell: 3x-2y=1$ after it has been Rotated -135° about (–2,3).

Solution According to the note, we choose any two arbitrary points, say (1, 1) and (–1, –2). Together with $(x_o, y_o) = (-2, 3)$ and $\theta = -135^\circ$, we get

$$R(1, 1) = (-2-2.5\sqrt{2},\ 3-0.5\sqrt{2}) \quad \text{and} \quad R(-1,-2) = (-2-3\sqrt{2},\ 3+2\sqrt{2})$$

$$\Rightarrow \text{The slope of } \ell' = \frac{3+2\sqrt{2}-3+0.5\sqrt{2}}{-2-3\sqrt{2}+2+2.5\sqrt{2}} = -5$$

$$\Rightarrow \ell' : \frac{y-3-2\sqrt{2}}{x+2+3\sqrt{2}} = -5$$

$$\Rightarrow \ell' : y-3-2\sqrt{2} = -5x-10-15\sqrt{2}$$

$$\Rightarrow \ell' : y+5x+7+13\sqrt{2} = 0$$

Exercise 8.8

1 Rectangle ABCD has vertices A (1, 2), B (4, 2), C (4, –1) and D (1, –1).

Find the images of the vertices of the rectangle when the axes are rotated about the origin through an angle $\theta = \pi$

2 Find the point into which the given point is transformed by a rotation of the axes through the indicated angles, about the origin.

a (–3, 4); 90° **b** (–2, 0); 60° **c** $(0, -1); \frac{\pi}{4}$ **d** (–1, 2); 30°

3 Find an equation of the line into which the line with the given equation is transformed under a rotation through the indicated angle.

a $3x - 4y = 7$; acute angle θ such that $\tan \theta = \frac{3}{4}$

b $2x + y = 3$; $\theta = \frac{\pi}{3}$

4 Find an equation of the circle into which the circle with the given equation is transformed under a rotation through the indicated angle, about the origin.

a $x^2 + y^2 = 1$, $\theta = \frac{\pi}{3}$ **b** $(x + 1)^2 + (y - 2)^2 = 3^2$, $\theta = \frac{\pi}{4}$

5 Find the image of (1, 0) after it has been rotated –60° about (3, 2).

6 If M is a reflection in the line $y = -x$ and R is a rotation about the origin through 90°, find

a M(R(3, 0)) **b** R(M(3, 0))

7 In a rotation R, the image of A(6, 2) is A′(3, 5) and the image of B(7, 3) is B′(2, 6). Find the image of (0, 0).

8 In Figure 8.38, point B is the image of point A in a reflection about the line ℓ and point C is the image of point B in a reflection about the line t. Prove that there is a rotation about O through an angle 2θ that will map C to A.

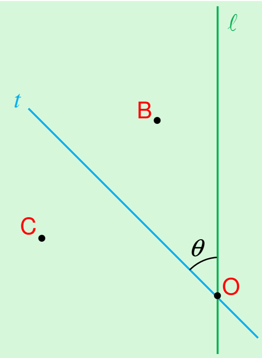


Figure 8.38

Key Terms

coordinate form of a vector
identity transformation
initial point
non-rigid motion
parallel vectors
perpendicular (orthogonal) vectors
reflection
resolution of vectors
rigid motion
rotation
scalar quantity
standard position
standard unit vector
terminal point
transformation
translation
unit vector
vector quantity
zero vector

Summary

1 *Vector*

i A quantity which can be completely described by its magnitude expressed in some particular unit is called a **scalar quantity**.

ii A quantity which can be completely described by stating both its magnitude expressed in some particular unit and its direction is called a **vector quantity**.

iii Two vectors are said to be **equal**, if they have the same magnitude and direction.

iv A **zero vector** or **null vector** is a vector whose magnitude is zero and whose direction is indeterminate.

v A **unit vector** is a vector whose magnitude is one.

2 ***Addition of vectors***

Let **u** and **v** be vectors, then the sum **u** + **v** is a vector given by the parallelogram law or triangle law satisfying the following properties.

- **i** Vector addition is commutative. **u** + **v** = **v** + **u**
- **ii** Vector addition is associative. (**u** + **v**) + **w** = **u** + (**v** + **w**)
- **iii** $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- **iv** $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- **v** $|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$

3 ***Multiplication of a vector by a scalar***

Let **u** be a vector and λ be a scalar, then $\lambda\mathbf{u}$ is a vector satisfying the following properties.

- **i** $|\lambda\mathbf{u}| = |\lambda||\mathbf{u}|$
- **ii** If μ is a scalar, then $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$
- **iii** If **v** is a vector, then $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$.

4 ***Scalar product or dot product***

The dot product of two vectors, **u** and **v** and θ is an angle between them is defined as: $\mathbf{u}.\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ satisfying the following properties.

- **i** The scalar product of vectors is commutative. **u**.**v** = **v**.**u**.
- **ii** If **u** = **0** or **v** = **0**, then **u**.**v** = 0.
- **iii** Two vectors **u** and **v** are orthogonal if **u.v** = 0.

5 ***Transformation of the plane***

- **i** Transformation can be classified as **rigid** motion and non-rigid motion.
- **ii** **Rigid motion** is a motion that preserves distance. Otherwise it is non-rigid.
- **iii** **Identity transformation** is a transformation that image of every point is itself.

6 ***Translation***

Translation is a transformation in which every point of a figure is moved along the same direction through the same distance.

- **i** **Translation vector:** If point P is translated to P', the vector $\overrightarrow{PP'}$ is said to be the translation vector.
- **ii** If **u** = (h, k) is a translation vector, then T(x, y)= ($x + h$, $y + k$)

7 ***Reflection***

A reflection M about a fixed line L is a transformation of the plane onto itself which maps each point P of the plane into the point P' of the plane such that L is the perpendicular bisector of PP'.

i Reflection in the x-axis, $M(x, y) = (x, -y)$

ii Reflection in the y-axis, $M(x, y) = (-x, y)$

iii Reflection in the line $y = x$, $M(x, y) = (y, x)$

iv Reflection in the line $y = -x$, $M(x, y) = (-y, -x)$

v Reflection in the line $y = mx$, $M(x, y) = (x', y')$

$x' = x \cos 2\theta + y \sin 2\theta \qquad y' = x \sin 2\theta - y \cos 2\theta$

$m = \tan\theta$

8 ***Rotation***

A rotation R about a point O through an angle θ is a transformation of the plane onto itself which maps every point P of the plane into the point P' of the plane such that $OP = OP'$ and $m(\angle POP') = \theta$

Rotation formulae

$x' = x \cos\theta - y \sin\theta \qquad y' = x \sin\theta + y \cos\theta$

Review Exercises on Unit 8

1 Given vectors $\mathbf{u} = (2, 5)$, $\mathbf{v} = (-3, 3)$ and $\mathbf{w} = (5, 3)$

A Find $\mathbf{u} - \mathbf{v} + 2\mathbf{w}$ and $|\mathbf{u} - \mathbf{v} + 2\mathbf{w}|$ **b** Find $2\mathbf{u} + 3\mathbf{v} - \mathbf{w}$ and $|2\mathbf{u} + 3\mathbf{v} - \mathbf{w}|$

c Find the unit vector in the direction of **u**.

d Find **z** if $\mathbf{z} + \mathbf{u} = \mathbf{v} - \mathbf{w}$ **e** Find **z** if $\mathbf{u} + 2\mathbf{z} = 3\mathbf{v}$

2 Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ with $|\mathbf{F}_1| = 30$N and $|\mathbf{F}_2| = 40$N act on a point, if the angle between $\mathbf{F}_1$ and $\mathbf{F}_2$ is 30^o, then find the magnitude of the resultant force.

3 A rotation R takes A (1, –3) to A' (3, 5) and B (0, 0) to B' (4, –6). Find the centre of rotation.

4 If **a** and **b** are non-zero vectors with $|\mathbf{a}| = |\mathbf{b}|$, show that $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are orthogonal.

5 A person pulls a body 50 m on a horizontal ground by a rope inclined at 30^o to the ground. Find the work done by the horizontal component of the tension in the rope, if the magnitude of the tension is 10 N.

6 Using vector methods, find the equation of the line tangent to the circle

$x^2 + y^2 - x + y = 6$ at

a A(1, –3) **b** B(1, 2).

7 If a translation T carries the point (7, –12) to (9,–10), find the images of the following lines and circles.

a $y = 2x - 5$ **b** $2y - 5x = 4$ **c** $x + y = 10$

d $x^2 + y^2 = 3$ **e** $x^2 + y^2 - 2x + 5y = 0$

8 In a reflection, the image of the point P (3, 10) is P' (7, 2). Find the equation of the line of reflection.

9 If the plane is rotated 30^o about (1, 4) find the image of

a the point (–3, 2) **b** $x^2 + y^2 - 2x - 8y = 10$

c $x^2 + y^2 - 3y = 0$ **d** $y = x + 4$

10 Prove that the sum of all vectors from the centre of a regular polygon to each side is **0**.

11 Using a vector method, prove that an angle inscribed in a semi-circle measures 90^o.

12 Find the resultant of two vectors of magnitudes 6 units and 10 units, if the angle between them is:

a 30^o **b** 120^o **c** 150^o

13 Four forces acting on a particle are represented by 3**i** + 4**j**, 3**i** – 5**j**, 5**i** + 4**j** and 2**i** + **j**. Find the resultant force **F**.

14 A balloon is rising 4 meters per second. If a wind is blowing horizontally at a speed of 2.5 meter per second, find the velocity of the balloon relative to the ground.

15 Three towns A, B and C are joined by straight railways. Town B is 600km east and 1200km north of town A. Town C is 800 km east and 900 km south of town B. By considering town A as the origin,

a find the position vectors of B and C using the unit vectors **i** and **j**.

b if T is a train station two thirds of the way along the rail way form town A to town B, prove that T is the closest station to town C on the rail way from town A to town B.

16 Two villages A and B are 2 km and 4 km far away from a straight road respectively as shown is Figure 8.39.

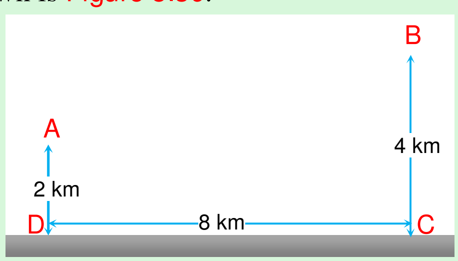
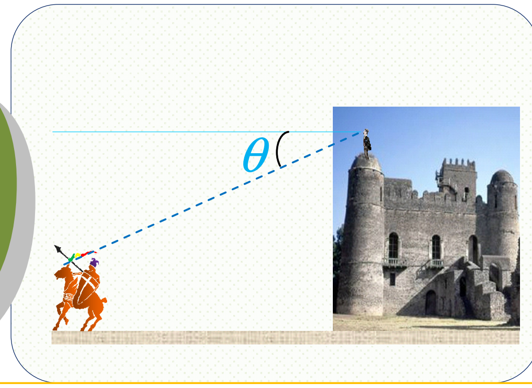


Figure 8.39

The distance between C and D is 8 km. Indicate the position of a common power supplier that is closest to both villages. Determine the sum of the minimum distances from the power supplier to both villages.

Unit 9



FURTHER ON TRIGONOMETRIC FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *know basic concepts about reciprocal functions.*
- *sketch graphs of some trigonometric functions.*
- *apply trigonometric functions to solve related problems.*

Main Contents:

9.1 THE FUNCTIONS Y = SEC X, Y = COSEC X AND Y = COT X

9.2 INVERSE OF TRIGONOMETRIC FUNCTIONS

9.3 GRAPHS OF SOME TRIGONOMETRIC FUNCTIONS

9.4 APPLICATION OF TRIGONOMETRIC FUNCTIONS

Key Terms

Summary

Review Exercises

INTRODUCTION

Trigonometry is the branch of mathematics that studies the relationship between angles and sides of a triangle. The values of the basic trigonometric functions are the ratio of the lengths of the sides of right-angled triangles.

Although "trigonometry" originated as the study of calculating the angles and lengths in triangles, it has much more widespread applications.

One of the earliest known uses of trigonometry is an Egyptian table that shows the relationship between the time of day and the length of the shadow cast by a vertical stick. The Egyptians knew that this shadow was longer in the morning, decreased to a minimum at noon, and increased thereafter until sun-down. The rule that gives time of day as a function of shadow length is a forerunner of the tangent and cotangent functions (trigonometric functions) you study in this unit.

9.1 THE FUNCTIONS *y* = sec x, *y* = cosec x AND *y* = cot x

You have learnt that the three fundamental trigonometric functions of the acute angle θ are defined as follows.

Name of Function	Abbreviation	Value at θ
sine	sin	$\sin\theta = \frac{opp}{hyp}$
cosine	cos	$\cos\theta = \frac{adj}{hyp}$
tangent	tan	$\tan\theta = \frac{opp}{adj}$

Considering the standard right-angled triangle and looking at the ratios these basic trigonometric functions represent in relation to angle A, you obtain:

$$\sin A = \frac{opposite}{hypotenuse} = \frac{a}{c}$$

$$\cos A = \frac{adjacent}{hypotenuse} = \frac{b}{c}$$

$$\tan A = \frac{opposite}{adjacent} = \frac{a}{b}$$

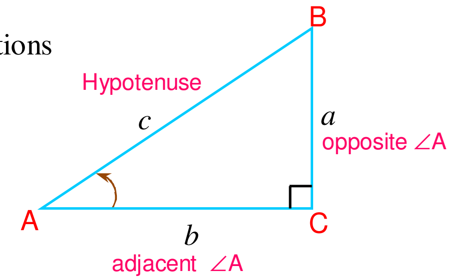
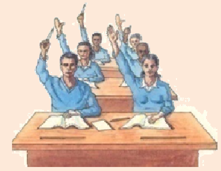


Figure 9.1

ACTIVITY 9.1



1 Given the triangle in Figure 9.2 below, find

a $\sin A$ **b** $\sin B$ **c** $\cos B$ **d** $\tan B$

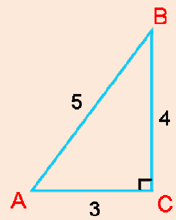


Figure 9.2

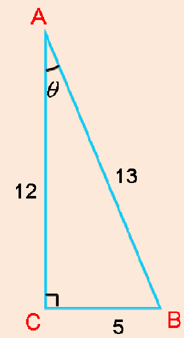


Figure 9.3

2 Given the triangle in Figure 9.3 above, evaluate

a $\dfrac{1}{\sin\theta}$ **b** $\dfrac{1}{\cos\theta}$ **c** $\dfrac{1}{\tan\theta}$

There are actually six trigonometric functions. The reciprocals of the ratios that define the sine, cosine and tangent functions are used to define the remaining three trigonometric functions. These reciprocal functions of the acute angle θ are defined as follows.

Name of Function	Abbreviation	Value at θ
cosecant	csc	$\csc\theta = \dfrac{hyp}{opp}$
secant	sec	$\sec\theta = \dfrac{hyp}{adj}$
cotangent	cot	$\cot\theta = \dfrac{adj}{opp}$

The relationship of these trigonometric functions in a standard right angled triangle is shown below.

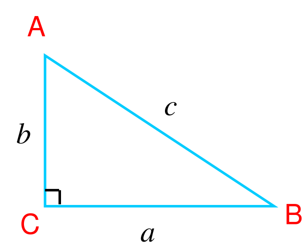


Figure 9.4

$$\csc A = \frac{c}{a} = \frac{1}{\sin A}$$

$$\sec A = \frac{c}{b} = \frac{1}{\cos A}$$

$$\cot A = \frac{b}{a} = \frac{1}{\tan A}$$

Example 1 Given the triangle below, find:

a $\cot A$ **b** $\csc B$

c $\sec A$ **d** $\csc A$

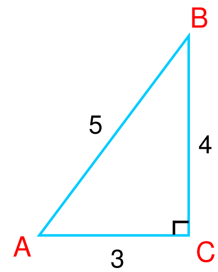


Figure 9.5

Solution

a $\cot A = \frac{3}{4}$ **b** $\csc B = \frac{5}{3}$

c $\sec A = \frac{5}{3}$ **d** $\csc A = \frac{5}{4}$

Graphs of $y = \csc x$, $y = \sec x$ and $y = \cot x$

In Grade 10, you studied the graphs of the sine, cosine and tangent functions. In this topic you will study the graphs of the remaining three trigonometric functions.

Group Work 9.1

1 Determine the domain, range and period for the following three trigonometric functions and draw their graphs.

a $y = \sin x$ **b** $y = \cos x$ **c** $y = \tan x$

2 Based on your knowledge of trigonometric functions, fill in the following table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\csc\theta$							
$\sec\theta$							
$\cot\theta$							

3 Determine the domain of

a $y = \csc x$ **b** $y = \sec x$ **c** $y = \cot x$

4 You know that $-1 \le \sin x \le 1$ for all $x \in \mathbb{R}$. In short, $|\sin x| \le 1$; what can you say about $\frac{1}{|\sin x|}$?

5 You also know that $\frac{1}{\sin(x+2\pi)} = \frac{1}{\sin x} = \csc x$.

Are $\csc x$, $\sec x$ and $\cot x$ periodic? If your answer is yes, determine their periods.

6 Discuss the symmetric properties of secant, cosecant and cotangent functions.

From Group work 9.1, you should have determined the domain, range and period of the cosecant, secant and cotangent functions as follows.

1 If $f(x) = \csc x$, then $D_f = \{x \in \mathbb{R} : x \neq k\pi, k \in \mathbb{Z}\}$

$$\text{Range} = (-\infty, -1] \cup [1, \infty)$$

$$\text{Period, } P = 2\pi$$

2 If $f(x) = \sec x$, then $D_f = \left\{x \in \mathbb{R} : x \neq \frac{(2k+1)\pi}{2}; k \in \mathbb{Z}\right\}$

$$\text{Range} = (-\infty, -1] \cup [1, \infty)$$

$$\text{Period, } P = 2\pi$$

3 If $f(x) = \cot x$, then $D_f = \{x \in \mathbb{R} : x \neq k\pi, k \in \mathbb{Z}\}$

$$\text{Range} = \mathbb{R}$$

$$\text{Period, } P = \pi$$

You now want to draw the graph of

$$f(x) = \csc x$$

The domain of cosecant function is restricted, in order to have no division by zero. By taking the reciprocals of non-zero ordinates on the graph of the sine function as shown in Figure 9.6, you obtain the graph of $f(x) = \csc x$.

The graph of cosecant function has vertical asymptotes at the point where the graph of the sine function crosses the x-axis.

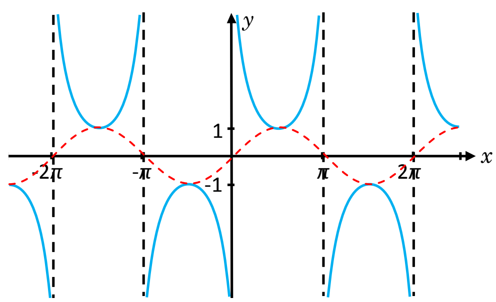


Figure 9.6 *Graph of y = csc x*

Applying the same techniques as for the cosecant function, we can draw the graphs of secant and cotangent functions as follows.

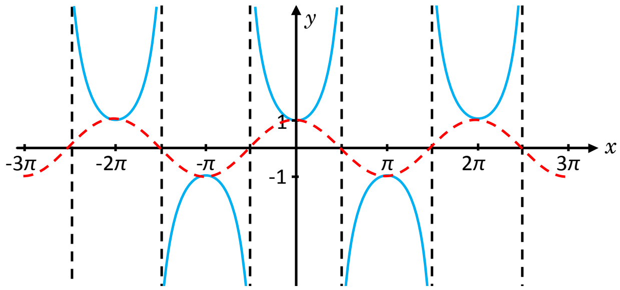


Figure 9.7 *Graph of y = sec x*

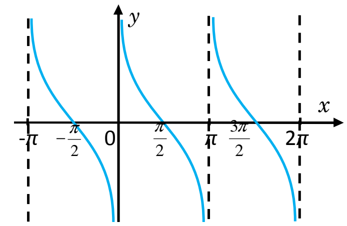


Figure 9.8 *Graph of y = cot x*

Exercise 9.1

1 Determine each of the following values without the use of tables or calculators.

a $\sec\left(\frac{\pi}{4}\right)$ **b** $\csc\left(-\frac{\pi}{2}\right)$ **c** $\cot\left(\frac{-3\pi}{4}\right)$

d $\sec\left(\frac{\pi}{3}\right)$ **e** $\csc\left(-\frac{\pi}{6}\right)$ **f** $\cot\left(\frac{5\pi}{6}\right)$

g $\sec\left(\frac{2\pi}{3}\right)$ **h** $\csc\left(\frac{7\pi}{3}\right)$ **i** $\cot\left(\frac{7\pi}{6}\right)$

j $\cot(-\pi)$ **k** $\sec\left(\frac{5\pi}{2}\right)$ **l** $\csc(3\pi)$

2 Determine the largest interval $I \subseteq [0, 2\pi]$ on which

a $f(x) = \csc x$ is increasing. **b** $f(x) = \sec x$ is increasing.

c $f(x) = \cot x$ is increasing.

3 Simplify each of the following expressions.

a $\sec x \sin x$ **b** $\tan x \csc x$ **c** $1+\frac{\tan x}{\cos x}$

d $\csc\left(x+\frac{\pi}{2}\right)$ **e** $\sec\left(x-\frac{\pi}{2}\right)$ **f** $\tan\left(x+\frac{\pi}{2}\right)$

4 Find the range of $y = -3\sec x$.

5 Prove each of the following trigonometric identities.

a $\sec^2 x - \tan^2 x = 1$ **b** $\csc^2 x - \cot^2 x = 1$

9.2 INVERSE OF TRIGONOMETRIC FUNCTIONS

You now need to define inverses of the trigonometric functions, starting with a brief review of the general concept of inverse functions. You first restate a few important facts about inverse functions.

Facts about inverse functions

For f a one-to-one function and f^{-1} its inverse:

1. If (a, b) is an element of f, then (b, a) is an element of f^{-1}, and conversely.
2. Range f = Domain of f^{-1}
3. Domain of f = Range of f^{-1}

The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

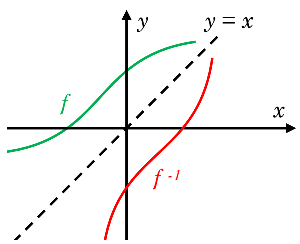
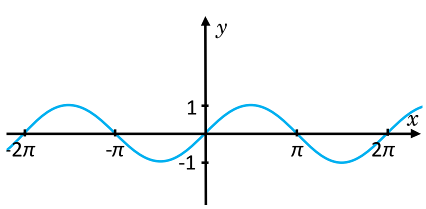
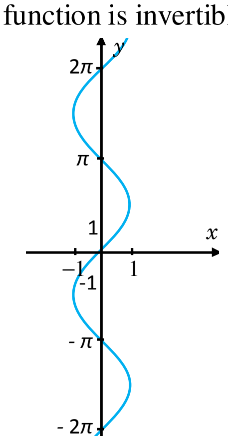


Figure 9.9

You know that a function f is invertible if it is one -to-one. All trigonometric functions are periodic; hence, each range value can be associated with infinitely many domain values. As a result, no trigonometric function is one-to-one. So without restricting the domains, no trigonometric function has an inverse function as shown in Figure 9.10 below. To resolve this problem, you restrict the domain of each function so that it is one-to-one over the restricted domain. Thus, for this restricted domain, the function is invertible.





a *Graph of $y = \sin x$*

b *Graph of $y = \sin^{-1} x$ on domain = $[-1, 1]$ and range $(-\infty, \infty)$*

Figure 9.10

Inverse trigonometric functions are used in many applications and mathematical developments and they will be particularly useful to you when you solve trigonometric equations.

ACTIVITY 9.2

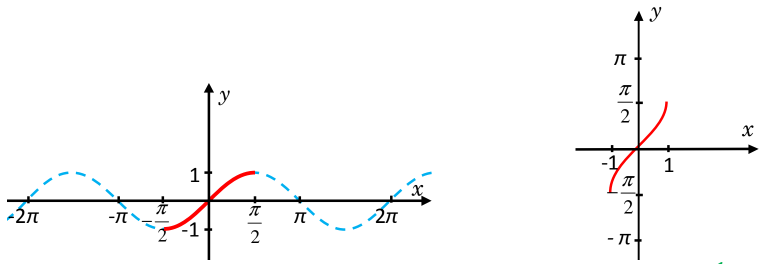
1 Find some intervals on which the sine function is one-to-one.

2 Draw the graph of $f(x) = \sin x$ when $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and reflect it in the line $y = x$.

a Inverse sine function

From Activity 9.2, you should have seen that the sine function is invertible on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Now, you can define the inverse sine function as follows.



a *Graph of y = sin x* **b** *Graph of y = sin⁻¹ x*

Figure 9.11

Definition 9.1 Inverse sine or Arcsine function

The inverse sine or arcsine function, denoted by **sin⁻¹** or **arcsin**, is defined by

$\sin^{-1}x = y$ or $\arcsin x = y$, if and only if $x = \sin y$ for $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$

☑ Remark:

1 The inverse sine function is the function that assigns to each number x in $[-1, 1]$ the unique number y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $x = \sin y$.

2 Domain of $\sin^{-1} x$ is $[-1, 1]$ and Range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

3 From the definition, you have

$\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ $\qquad$ $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

☑ Caution:

$\sin^{-1} x$ is different from $(\sin x)^{-1}$ and $\sin x^{-1}$;

$(\sin x)^{-1} = \dfrac{1}{\sin x}$ and $\sin x^{-1} = \sin\left(\dfrac{1}{x}\right)$

Example 1 Calculate $\sin^{-1} x$ for

a $x = 0$ **b** $x = 1$ **c** $x = \dfrac{\sqrt{3}}{2}$ **d** $x = -1$

Solution

a $\sin^{-1}(0) = 0$ since $\sin 0 = 0$ and $0 \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

b $\sin^{-1}(1) = \dfrac{\pi}{2}$ since $\sin\left(\dfrac{\pi}{2}\right) = 1$ and $\dfrac{\pi}{2} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

c $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3}$ since $\sin \dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$ and $\dfrac{\pi}{3} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

d $\sin^{-1}(-1) = -\dfrac{\pi}{2}$ since $\sin\left(-\dfrac{\pi}{2}\right) = -1$ and $-\dfrac{\pi}{2} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

Example 2 Compute $\cos\left(\sin^{-1}\left(\dfrac{4}{7}\right)\right)$.

Solution Let $\theta = \sin^{-1}\left(\dfrac{4}{7}\right)$. Then, $\sin\theta = \dfrac{4}{7}$ and drawing the reference triangle associated with θ, you have:

$$\cos\theta = \frac{\sqrt{33}}{7}$$

where $\sqrt{33}$ is calculated using **Pythagoras' theorem**.

Therefore, $\cos\left(\sin^{-1}\left(\dfrac{4}{7}\right)\right) = \cos\theta = \dfrac{\sqrt{33}}{7}$

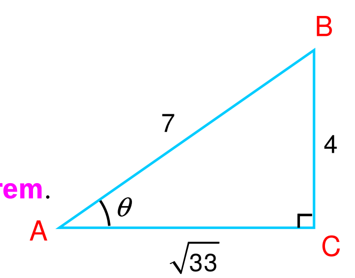


Figure 9.12

Calculator Tips

Read the user's manual for your calculator and find the values to 4 significant digits for

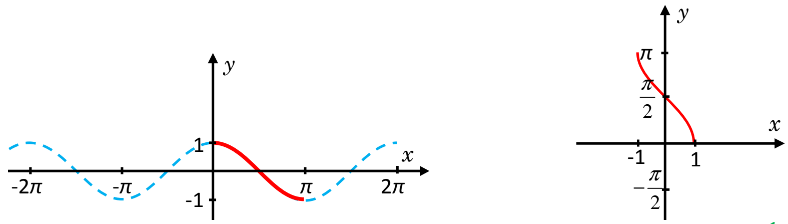
1 arcsin (0. 0215)

2 $\sin^{-1}(-0.137)$

3 $\tan(\sin^{-1}(0.9415))$

b Inverse cosine function

You know that cos x is not one-to-one. Note, however, that cos x decreases from 1 to –1 in the interval $[0, \pi]$. Thus if y = cos x and x is restricted in the interval $[0, \pi]$, then for every y in $[-1, 1]$, there is a unique x such that cos x = y.



a *Graph of y = cos x* **b** *Graph of y = cos^-1 x*

Figure 9.13

Use this restricted cosine function to define the inverse cosine function. Reflecting the graph of $y = \cos x$ on $[0, \pi]$ in the line $y = x$, gives the graph of $f(x) = \cos^{-1} x$ as shown in Figures 9.13 and 9.14.

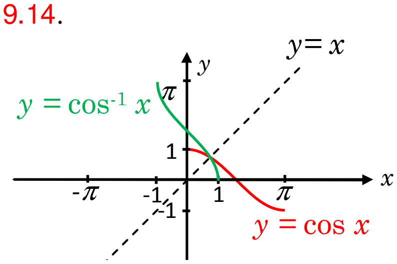


Figure 9.14

Definition 9.2

The **inverse cosine** or **arccosine** function, denoted by $\cos^{-1}$ or arccos, is defined by

$\cos^{-1} x = y$, if and only if $x = \cos y$, for $0 \le y \le \pi$.

☑ Remark:

1 Domain of $\cos^{-1}x$ is $[-1, 1]$ and Range of $\cos^{-1}x$ is $[0, \pi]$

2 From the definition, you have

$\cos(\cos^{-1}x) = x$, if $-1 \le x \le 1$.

$\cos^{-1}(\cos x) = x$, if $0 \le x \le \pi$.

Example 3 Calculate $y = \cos^{-1} x$ for

a $x = 0$ **b** $x = 1$ **c** $x = \dfrac{\sqrt{3}}{2}$ **d** $x = -1$

Solution

a $\cos^{-1}(0) = \dfrac{\pi}{2}$ since $\cos\dfrac{\pi}{2} = 0$ and $\dfrac{\pi}{2} \in [0, \pi]$

b $\cos^{-1}(1) = 0$ since $\cos 0 = 1$ and $0 \in [0, \pi]$

c $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ since $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{6} \in [0, \pi]$

d $\cos^{-1}(-1) = \pi$ since $\cos \pi = -1$ and $\pi \in [0, \pi]$

Example 4 Compute $\tan\left(\cos^{-1}\left(\frac{1}{4}\right)\right)$

Solution Let $\theta = \cos^{-1}\left(\frac{1}{4}\right)$, so that $\cos\theta = \frac{1}{4}$.

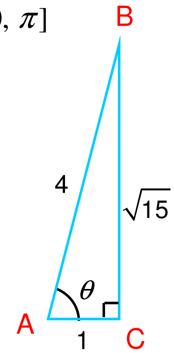


Figure 9.15

The opposite side is $\sqrt{4^2 - 1^2} = \sqrt{15}$

Thus, $\tan\left(\cos^{-1}\left(\frac{1}{4}\right)\right) = \tan\theta = \frac{\sqrt{15}}{1} = \sqrt{15}$

Example 5 Show that $\cos^{-1}(-x) = \pi - \cos^{-1} x$.

Solution Let $y = \pi - \cos^{-1} x$. Then, $\cos^{-1} x = \pi - y$

$\Rightarrow \quad x = \cos(\pi - y) \Rightarrow x = -\cos y \Rightarrow -x = \cos y$

$\Rightarrow \quad \cos^{-1}(-x) = y = \pi - \cos^{-1} x$

Calculator Tips

Find to 4 significant digits

1. arccos (0.5214)
2. $\cos^{-1}(-0.0103)$
3. sec (arccos (0.04235))

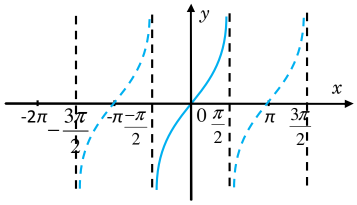
Example 6 Compute $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Solution $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \pi - \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$

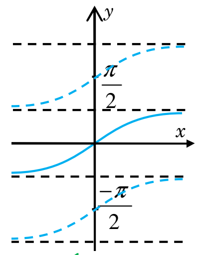
c Inverse tangent function

The function $\tan x$ is not one-to-one on its domain as it can be seen from its graph.

To get a unique x for a given y, you restrict x to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.



a *Graph of y = tan x*



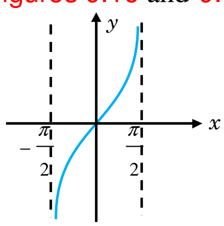
b *Graph of $y = \tan^{-1} x$ and range = $\{y \in \mathbb{R}: y \neq (2n-1)\frac{\pi}{2}; n \in \mathbb{N}\}$.*

Figure 9.16

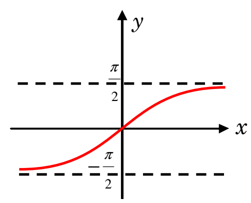
Definition 9.3

The **inverse tangent function** is a function denoted by $\tan^{-1}x$ or arctan x that assigns to each real number x the unique number y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $x = \tan y$.

Reflecting the graph of $y = \tan x$ in the line $y = x$ gives the graph of $f(x) = \tan^{-1} x$ as shown in the Figures 9.16 and 9.17.



a *Graph of y = tan x*



b *Graph of $y = \tan^{-1} x$*

Figure 9.17

☑ Remark:

1 Domain of $\tan^{-1}x$ is $(-\infty, \infty)$ and Range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

You stress that $\frac{\pi}{2}$ is not in the range of $\tan^{-1}x$ because $\tan \frac{\pi}{2}$ is not defined.

2 From the above definition, you have,

$\tan(\tan^{-1} x) = x$ for all real x

$\tan^{-1}(\tan x) = x$, if $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Example 7 Compute (in radians).

a $\tan^{-1}(0)$ **b** $\tan^{-1}\left(\sqrt{3}\right)$ **c** $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution

a $\tan^{-1}(0) = 0$ because $\tan(0) = 0$ and $0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$ because $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

c $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ because $\tan -\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$ and $-\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Example 8 Express $\tan(\sin^{-1} x)$ in terms of x.

Solution Here, you consider the following cases.

i Suppose $x = 0$, then $\tan(\sin^{-1}(0)) = \tan 0 = 0$.

ii Suppose $0 < x < 1$. Let $\theta = \sin^{-1} x$, then $\sin\theta = x$ and $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

Look at the reference triangle given.

Hence, $\tan(\sin^{-1} x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}$

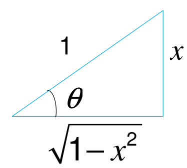


Figure 9.18

iii If $-1 < x < 0$, then $\tan(\sin^{-1}x) = -\tan(\sin^{-1}(-x))$

$\Rightarrow \tan(\sin^{-1}(x)) = -\frac{-x}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}$

$\therefore \tan(\sin^{-1}(x)) = \frac{x}{\sqrt{1-x^2}}$ for all $x \in (-1, 1)$

Inverse cotangent, secant, and cosecant functions

Here, the definitions of the inverse cotangent, secant and cosecant functions are given. Whereas drawing the graphs is given as exercise.

Definition 9.4

i The **inverse cotangent** function $\cot^{-1}$ or arccot is defined by

$y = \cot^{-1}x$, if and only if $x = \cot y$ where $0 < y < \pi$ and $-\infty < x < \infty$.

ii The **inverse secant function** $\sec^{-1} x$ or arcsec x is defined by

$y = \sec^{-1}x$, if and only if $x = \sec y$ where $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$, $|x| \ge 1$.

iii The **inverse cosecant function** $\csc^{-1} x$ or arccsc x is defined by

$y = \csc^{-1}x$, if and only if $x = \csc y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$, $|x| \ge 1$.

Example 9 Find the exact values of

a $\cot^{-1}\left(\sqrt{3}\right)$ **b** $\sec^{-1}(2)$ **c** $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

Solution

a $y=\cot^{-1}\left(\sqrt{3}\right) \Rightarrow \cot y=\sqrt{3}$ and $0<y<\pi \Rightarrow y=\frac{\pi}{6}$

b $\sec^{-1}(2)=\frac{\pi}{3}$ because $\sec\left(\frac{\pi}{3}\right)=2$ and $0<\frac{\pi}{3}<\frac{\pi}{2}$

c $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)=\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}$

Exercise 9.2

1 Find the exact values of each of the following expressions without using a calculator or tables.

a $\sin^{-1}\left(-\frac{1}{2}\right)$ **b** $\cos^{-1}(3)$ **c** $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

d $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$ **e** $\sec^{-1}\left(\sqrt{2}\right)$ **f** $\cot^{-1}(-1)$

g $\cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$ **h** $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)$ **i** $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$

j $\arccos\left(\cos\left(\frac{5\pi}{6}\right)\right)$ **k** $\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$ **l** $\tan\left(\tan^{-1}\sqrt{3}\right)$

m $\tan\left(\arcsin\left(\frac{\sqrt{3}}{2}\right)\right)$ **n** $\cos^{-1}\left(\tan\left(\frac{-\pi}{4}\right)\right)$

2 Express each of the following expressions in terms of x.

a $y = \sin(\arctan x)$ **b** $y = \cos(\arcsin x)$ **c** $y = \tan(\arccos x)$

3 Prove each of the following identities.

a $\tan^{-1}(-x) = -\tan^{-1} x$ **b** $\text{arcsec } x = \arccos\left(\frac{1}{x}\right)$ for $|x| \geq 1$

c $\sec^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$ for $|x| \geq 1$

4 Sketch the graph of:

a $y = \text{arccsc } x$ **b** $y = \text{arcsec } x$ **c** $y = \text{arccot } x$

5 Let $y = 3 + 2 \arcsin(5x - 1)$. Express x in terms of y and determine the range of values of x and y.

9.3 GRAPHS OF SOME TRIGONOMETRIC FUNCTIONS

In the previous section, the graphs of $y = \sin x$ and $y = \cos x$ have been discussed. In this section, you will consider graphs of the more general forms:

$$y = a \sin (kx + b) + c \text{ and } y = a \cos (kx + b) + c$$

These equations are important in both mathematics and related fields. They are used in the analysis of sound waves, *x*-rays, electric circuits, vibrations, spring-mass systems, etc.

Group Work 9.2

1 For the following values of x, fill in a table for the given functions.

x	$\sin x$	$2 \sin x$	$\cos x$	$-3 \cos x$	$\frac{2}{3}\cos x$
0	0	0	1	-3	$\frac{2}{3}$
$\frac{\pi}{6}$	$\frac{1}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{-3\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$					
. . .					
2π					

Copy and complete the table for

$$x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{4\pi}{3}, \frac{7\pi}{4}, \frac{11\pi}{6}, 2\pi$$

2 Using the above table, sketch the graphs of the following pairs of functions using the same coordinate axes.

a $y = \sin \theta$ and $y = 2 \sin\theta$

b $y = \sin \theta$ and $y = \frac{1}{2} \sin \theta$

c $y = \cos \theta$ and $y = -3 \cos \theta$

d $y = \cos \theta$ and $y = \frac{2}{3} \cos \theta$

3 For each of the following functions, find the ranges and periods.

a $y = 2 \sin x$ **b** $y = \frac{1}{2} \sin x$

c $y = -3 \cos x$ **d** $y = \frac{2}{3} \cos x$

4 Let $a \in \mathbb{R}$; express the range of $y = a \cos x$ in terms of a. Here, $|a|$ is said to be the amplitude of $a \cos x$. In general, if f is a periodic function, the amplitude of f is given by

$$|a| = \frac{\text{Maximum value of } f - \text{Minimum value of } f}{2}.$$

Find the amplitudes of each of the following trigonometric functions.

a $f(x) = \sin x$ **b** $g(x) = -\cos x$ **c** $h(x) = 0.25 \sin x$

d $k(x) = 4 \tan x$ **e** $s(x) = -6 \cos x$ **f** $f(x) = |\sin x|$

From Group Work 9.2 you should have observed that the graph of $y = a \sin x$ can be obtained from the graph of $y = \sin x$ by multiplying each value of y on the graph of $y = \sin x$ by a.

✓ The graph of $y = a \sin x$ still crosses the x-axis where the graph of $y = \sin x$ crosses the x-axis, because $a \times 0 = 0$.

✓ Since the maximum value of $\sin x$ is 1, the maximum value of $a \sin x$ is $|a| \times 1 = |a|$. The constant $|a|$, the **amplitude** of the graph of $y = a \sin x$, indicates the maximum deviation of the graph of $y = a \sin x$ from the x-axis.

✓ The period of $y = a \sin x$ is also 2π, since $a \sin(x + 2\pi) = a \sin x$.

Example 1 Draw the graphs of $y = \sin x$, $y = \frac{1}{2} \sin x$ and $y = -2 \sin x$, on the same coordinate system for $0 \le x \le 2\pi$.

Solution The amplitudes of $y = \frac{1}{2} \sin x$ and $y = -2 \sin x$ are $\frac{1}{2}$ and 2, respectively, and the amplitude of $\sin x$ is 1. The negative sign in $y = -2 \sin x$ reflects the graph of $y = 2 \sin x$ across the x-axis. Together with the results from Group Work 9.2, this gives you the graphs of all the three functions as shown in Figure 9.19a.

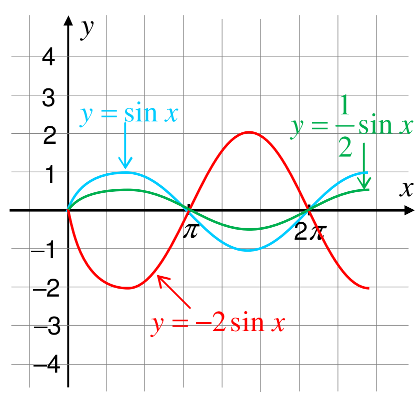


Figure 9.19 a

In general, for any function *f*, the graph of *af* is drawn by expanding or compressing the graph of *f* in the vertical direction and by reflecting in the *x*-axis when $a < 0$. For $a \neq \pm 1$, the amplitude of *af* is different from that of *f*, whereas the period doesn't change.

Similarly, the graphs of $y = \frac{1}{3}\cos x$, $y = -3\cos x$, $y = \cos x$, $0 \le x \le 2\pi$ are as shown in Figure 9.19b.

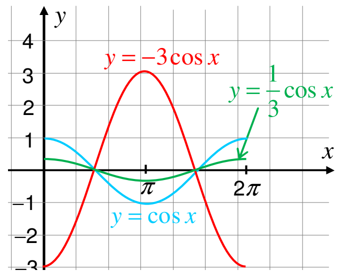


Figure 9.19b

9.3.1 The Graph of $f(x) = \sin kx$, $k > 0$

Group Work 9.3

1 Fill in the values of the following functions for the values of x given below.

x	$2x$	$\frac{1}{2}x$	$\sin(x)$	$\sin(2x)$	$\sin\left(\frac{1}{2}x\right)$
0					
$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{8}$	$\frac{\sqrt{2}}{2}$	1	
$\frac{\pi}{2}$					
⋮					
2π					

Copy and complete the table for

$$x = 0,\ \frac{\pi}{4},\ \frac{\pi}{2},\ \frac{3\pi}{4},\ \pi,\ \frac{5\pi}{4},\ \frac{3\pi}{2},\ \frac{7\pi}{4},\ 2\pi.$$

2 Find the maximum and minimum values of

a $f(x) = \sin(2x)$ **b** $g(x) = \sin\left(\frac{1}{2}x\right)$

3 Using the values in the table above, draw the graph of

a $f(x) = \sin(2x)$ **b** $g(x) = \sin\left(\frac{1}{2}x\right)$

From Group work 9.3, it can be observed that

- ✓ the function $f(x) = \sin(2x)$ covers one complete cycle on the interval $[0,\pi]$, but $g(x) = \sin\left(\frac{1}{2}x\right)$ covers exactly half of one cycle on the interval $[0, 2\pi]$.
- ✓ both functions are periodic and the shape of their graphs is a sine wave.

You can sketch the graphs of $f(x) = \sin(2x)$ and $g(x) = \sin\left(\frac{1}{2}x\right)$ based on these properties and some other strategic points such as the x and y-intercepts and the values of x which give minimum value or maximum value. Thus, for $0 \le x \le 2\pi$,

- $\sin(2x) = 0, \Rightarrow 2x = 0, \pi, 2\pi \Rightarrow x = 0, \frac{\pi}{2}, \pi$

$\Rightarrow$ The graph crosses the x-axis at $(0, 0)$, $\left(\frac{\pi}{2}, 0\right)$ and $(\pi, 0)$.

- $\sin(2x) = 1 \Rightarrow 2x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{4}$

$\Rightarrow$ The function attains its maximum value at $x = \frac{\pi}{4}$

- $\sin 2x = -1 \Rightarrow 2x = \frac{3\pi}{2} \Rightarrow x = \frac{3\pi}{4}$

$\Rightarrow$ The function attains its minimum value at $x = \frac{3\pi}{4}$

From all these, you have the following sketch of the curve of $f(x) = \sin(2x)$ drawn together with $y = \sin x$.

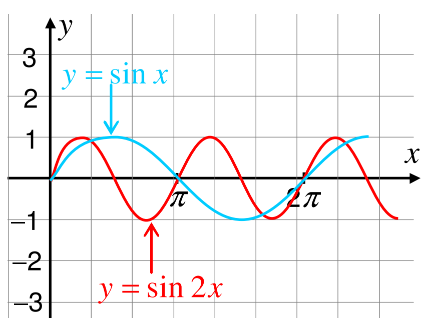


Figure 9.20

Note:

The period of $y = \sin(2x)$ is π. It has two complete cycles on $[0, 2\pi]$.

Similarly, for $0 \le x \le 4\pi$,

- $\sin\left(\frac{1}{2}x\right) = 0 \Rightarrow \frac{1}{2}x = 0,\ \pi,\ 2\pi$

$\Rightarrow\ x = 0, 2\pi, 4\pi$

$\Rightarrow$ The graph of $f(x) = \sin\left(\frac{1}{2}x\right)$ crosses the x-axis at (0, 0), (2π, 0) and (4π, 0).

- $\sin\left(\frac{1}{2}x\right) = 1 \Rightarrow \frac{1}{2}x = \frac{\pi}{2} \Rightarrow x = \pi$.

$\Rightarrow$ The graph has a peak at (π, 1)

- $\sin\left(\frac{1}{2}x\right) = -1 \Rightarrow \frac{1}{2}x = \frac{3}{2}\pi \Rightarrow x = 3\pi$

$\Rightarrow$ The graph has a valley at (3π, –1)

Based on the above facts, draw the graphs of $f(x) = \sin\left(\frac{1}{2}x\right)$ and $y = \sin x$ as follows

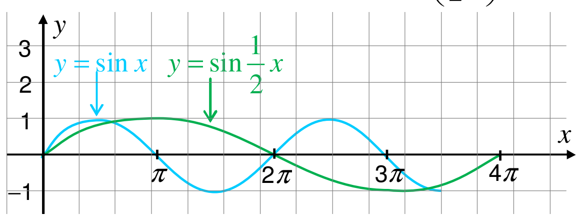


Figure 9.21

Now investigate the effect of k by comparing

$$y = \sin x \quad \text{and} \quad y = \sin(kx),\ k > 0$$

where both have the same amplitude, 1. Since $y = \sin x$ has period 2π, it follows that $y = \sin(kx)$ completes one cycle as kx varies from $kx = 0$ to $kx = 2\pi$ or as x varies from $x = 0$ to $x = \frac{2\pi}{k}$.

Thus, the period of $y = \sin(k\,x)$ is $\frac{2\pi}{k}$.

A similar investigation shows that the period of $y = \cos(kx)$ is $\frac{2\pi}{k}$.

If $k < 0$, remove the negative sign from inside the function by using the identities:

$$\sin(-x) = -\sin x \text{ and } \cos(-x) = \cos x.$$

In the case $k < 0$, the period of $y = \sin(k\,x)$ and $y = \cos(k\,x)$ is $\frac{2\pi}{|k|}$.

Graphs of $y = a \sin(kx)$ and $y = a \cos(kx)$

All the above discussions may lead you to the following procedures of drawing graphs.

Procedures for drawing graphs

Step 1: Determine the period $P = \frac{2\pi}{|k|}$ and the amplitude $|a|$.

Step 2: Divide the interval [0, P] along the x-axis into four equal parts:

$x = 0, \frac{P}{4}, \frac{P}{2}, \frac{3P}{4}, P$

Step 3: Draw the graph of the points corresponding to $x = 0, \frac{P}{4}, \frac{P}{2}, \frac{3P}{4}, P$.

x	0	$\frac{P}{4}$	$\frac{P}{2}$	$\frac{3P}{4}$	P
$a \sin(kx)$	0	a	0	$-a$	0
$a \cos(kx)$	a	0	$-a$	0	a

Step 4: Connect the points found in Step 3 by a sine wave.

Step 5: Repeat this one cycle of the curve as required.

Example 2 Draw the graph of $y = 2 \sin(3x)$.

Solution

Step 1: The period $P = \frac{2\pi}{3}$ and the amplitude $a = 2$.

Step 2: The curve completes one cycle on the interval $\left[0, \frac{2\pi}{3}\right]$.

Divide $\left[0, \frac{2\pi}{3}\right]$ into four equal parts by

$x = 0, \frac{P}{4} = \frac{\pi}{6}, \frac{P}{2} = \frac{\pi}{3}, \frac{3P}{4} = \frac{\pi}{2}, P = \frac{2\pi}{3}$

Step 3:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$2 \sin(3x)$	0	2	0	–2	0

Step 4: Connect the points found in Step 3 by a sine wave.

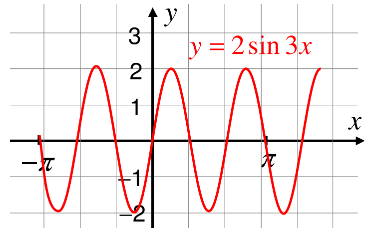


Figure 9.22

Example 3 Draw the graph of $f(x) = -3\cos\left(\frac{2}{3}x\right)$

Solution

Step 1: Period, $P = \frac{2\pi}{\left(\frac{2}{3}\right)} = 3\pi$ and amplitude, $|a| = |-3| = 3$

Step 2: Divide $[0, 3\pi]$ into four equal parts by

$$x = 0, \frac{P}{4} = \frac{3\pi}{4}, \frac{P}{2} = \frac{3\pi}{2}, \frac{3P}{4} = \frac{9\pi}{4}, P = 3\pi$$

Step 3:

x	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	3π
$-3\cos\left(\frac{2}{3}x\right)$	–3	0	3	0	–3

Step 4:

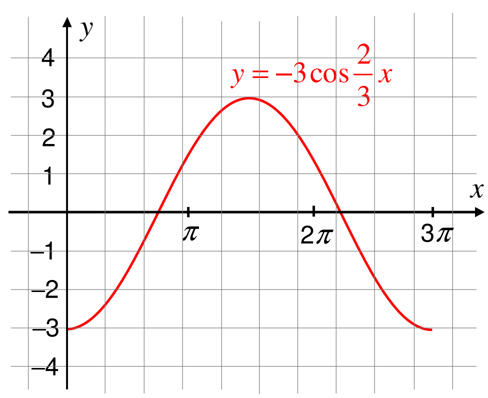


Figure 9.23

Exercise 9.3

1 Draw the graph of each of the following functions.

a $f(x) = 4\sin x$ **b** $f(x) = -2\cos x$

c $f(x) = \frac{2}{3}\sin x$ **d** $f(x) = \frac{1}{4}\cos x$

2 Draw the graph of each of the following trigonometric functions for one cycle. Indicate the amplitude and the period.

a $f(x) = \sin(4x)$ **b** $f(x) = -2\sin\left(\frac{1}{3}x\right)$

c $f(x) = \frac{2}{3}\cos(2x)$ **d** $f(x) = 5\sin\left(-\frac{2}{3}x\right)$

e $f(x) = 4\cos\left(\frac{1}{4}x\right)$ **f** $f(x) = \frac{1}{2}\cos\left(-\frac{3}{2}x\right)$

9.3.2 Graphs of $f(x) = a\sin(kx + b) + c$ and $f(x) = a\cos(kx + b) + c$

You have already sketched graphs of $f(x) = a\sin(kx)$ and $f(x) = a\cos(kx)$.

Here you are investigating the geometric effect of the constants b and c in drawing the graph of the functions.

Consider the function $y = a\sin(kx + b) + c$

$$\Rightarrow y - c = a\sin\left(k\left(x + \frac{b}{k}\right)\right)$$

This is simply the function $y = a\sin(kx)$ after it has been shifted $-\frac{b}{k}$ units in the x-direction and c units in the y-direction.

In particular, it is shifted to the positive x-direction if $\frac{b}{k} < 0$ and to the negative x-direction if $\frac{b}{k} > 0$. Also, it is shifted to the positive y-direction if $c > 0$ and to the negative y-direction if $c < 0$. For example, if you want to draw the graph of $y = 3\sin\left(2x - \frac{\pi}{3}\right) - 2$, rewrite the equation in the form

$$y + 2 = 3\sin\left(2\left(x - \frac{\pi}{6}\right)\right)$$

Thus, the graph of this function is obtained by shifting the graph of $y = 3\sin(2x)$ in the positive x-direction by $\frac{\pi}{6}$ units and 2 units in the negative y-direction as shown in Figure 9.22.

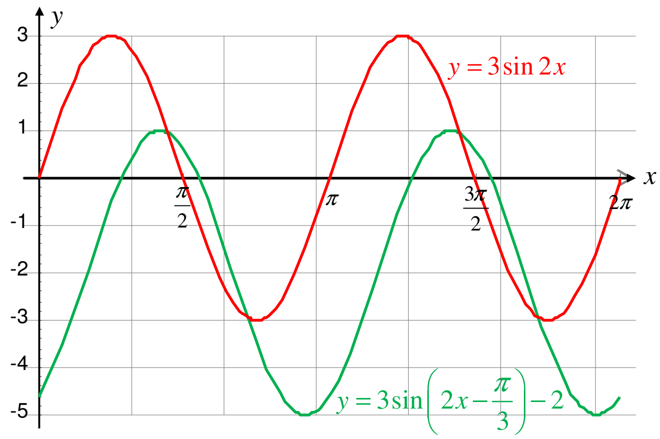
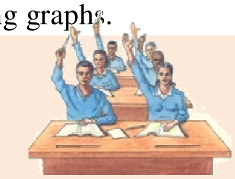


Figure 9.24

The following activity introduces a simplified procedure of drawing graphs.



ACTIVITY 9.3

1 If $3x + 5$ varies from 0 to 2π, i.e., $0 \le 3x+5 \le 2\pi$, then,

$$-5 \le 3x \le 2\pi - 5 \Rightarrow -\frac{5}{3} \le x \le \frac{2\pi}{3} - \frac{5}{3}\text{; i.e. } x \text{ varies from } \frac{-5}{3} \text{ to } \frac{2\pi}{3} - \frac{5}{3}$$

Based on this example, find the intervals on which x-varies, if each of the following expressions varies from 0 to 2π.

a $2x+1$ **b** $3x-1$ **c** $2x-\frac{\pi}{3}$ **d** $\pi x+\frac{\pi}{2}$

2 Find those values of x that divide the given interval into four equal parts.

a $[0, 2\pi]$ **b** $\left[\frac{1}{4}, \frac{\pi}{2} + \frac{1}{4}\right]$

3 Fill in the following table

x	$\frac{1}{4}$	$\frac{1}{4}+\frac{\pi}{8}$	$\frac{1}{4}+\frac{\pi}{4}$	$\frac{1}{4}+\frac{3\pi}{8}$	$\frac{1}{4}+\frac{\pi}{2}$
$3\sin(4x-1)$					
$3\cos(4x-1)$					

From Activity 9.3, you have the following properties.

1 If $kx + b$ varies from 0 to 2π, i.e., $0 \le kx + b \le 2\pi$, then,

$$-b \le kx \le -b + 2\pi \Rightarrow \frac{-b}{k} \le x \le \frac{-b}{k} + \frac{2\pi}{k} \quad (k > 0)$$

so that x varies from $\frac{-b}{k}$ to $\frac{-b}{k} + \frac{2\pi}{k}$

Therefore, $f(x) = \sin(kx + b)$ generates one cycle of sine wave as $kx + b$ varies from 0 to 2π, or as x varies over the interval $\left[\frac{-b}{k}, \frac{-b}{k} + \frac{2\pi}{k}\right]$.

2 The graph "starts" at $x = -\frac{b}{k}$ which is said to be the phase shift because the phase of the basic wave is shifted by a factor of b.

Furthermore, you have the following procedures for drawing graphs:

Assume that $k > 0$. (*If k < 0, use the symmetric properties of sine and cosine*).

Step 1: Determine the period, $P = \frac{2\pi}{k}$, the amplitude $= |a|$ and phase shift $= -\frac{b}{k}$

Step 2: Divide the interval $\left[\frac{-b}{k}, \frac{-b}{k} + \frac{2\pi}{k}\right]$ along the x-axis into four equal parts.

The length of each interval will be $\frac{\pi}{2k}$. Why? Explain!

The dividing values of x are:

$$x = \frac{-b}{k},\quad x = \frac{-b}{k} + \frac{\pi}{2k},\quad x = -\frac{b}{k} + \frac{\pi}{k},\quad x = \frac{-b}{k} + \frac{3\pi}{2k} \text{ and } x = \frac{-b}{k} + \frac{2\pi}{k}$$

Step 3: Draw the graph of the points corresponding to those values of x.

x	$-\frac{b}{k}$	$\frac{-b}{k} + \frac{\pi}{2k}$	$\frac{-b}{k} + \frac{\pi}{k}$	$\frac{-b}{k} + \frac{3\pi}{2k}$	$\frac{-b}{k} + \frac{2\pi}{k}$
$a \sin(kx + b)$	0	a	0	$-a$	0
$a \cos(kx + b)$	a	0	$-a$	0	a

Step 4: Connect the points found in Step 3 by a sine wave.

Step 5: Repeat this portion of the graph indefinitely to the left and to the right every $\frac{2}{k}\pi$ units on the x-axis.

Example 4 Draw the graph of $f(x) = 3\sin\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 1$.

Solution First draw the graph of $y = 3\sin\left(\frac{1}{2}x - \frac{\pi}{3}\right)$ and then shift it in the positive y-direction by 1 unit.

Step 1: The period, $P = \dfrac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$

Amplitude, $|a| = 3$

Phase shift, $\dfrac{-b}{k} = \dfrac{\frac{\pi}{3}}{\frac{1}{2}} = \dfrac{2\pi}{3}$

Step 2: $\left[\dfrac{-b}{k}, \dfrac{-b}{k} + \dfrac{2\pi}{k}\right] = \left[\dfrac{2\pi}{3}, \dfrac{2\pi}{3} + 4\pi\right] = \left[\dfrac{2\pi}{3}, \dfrac{14\pi}{3}\right]$.

The graph completes full cycle on $\left[\dfrac{2\pi}{3}, \dfrac{14\pi}{3}\right]$.

Divide $\left[\dfrac{2\pi}{3}, \dfrac{14\pi}{3}\right]$ into four equal parts by $x = \dfrac{2\pi}{3}, \dfrac{5\pi}{3}, \dfrac{8\pi}{3}, \dfrac{11\pi}{3}, \dfrac{14\pi}{3}$

Step 3:

x	$\frac{2\pi}{3}$	$\frac{5\pi}{3}$	$\frac{8\pi}{3}$	$\frac{11\pi}{3}$	$\frac{14\pi}{3}$
$3\sin\left(\frac{1}{2}x - \frac{\pi}{3}\right)$	0	3	0	–3	0

Step 4, 5:

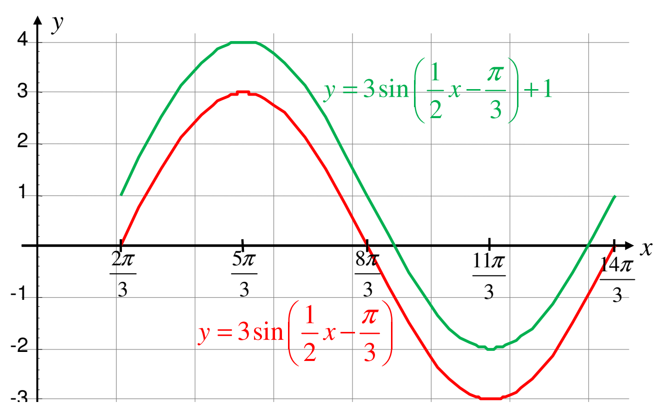


Figure 9.25

Example 5 Draw the graph of $f(x) = -5\cos(3x+2) - 2$.

Solution First draw the graph of $y = -5\cos(3x + 2)$ and then shift it in the negative y-direction by 2 units.

Step 1: Period, $P = \frac{2\pi}{3}$, amplitude, $|a| = |-5| = 5$.

Phase shift = $\frac{-2}{3}$, phase angle = –2

Step 2: Divide the interval $\left[\frac{-2}{3}, \frac{2}{3}\pi - \frac{2}{3}\right]$ into four equal intervals of length $\frac{\pi}{6}$.

Step 3:

x	$-\frac{2}{3}$	$\frac{-2}{3}+\frac{\pi}{6}$	$\frac{-2}{3}+\frac{\pi}{3}$	$\frac{-2}{3}+\frac{\pi}{2}$	$-\frac{2}{3}+\frac{2\pi}{3}$
$-5\cos(3x+2)$	–5	0	5	0	–5
$-5\cos(3x+2)-2$	–7	–2	3	–2	–7

Step 4, 5

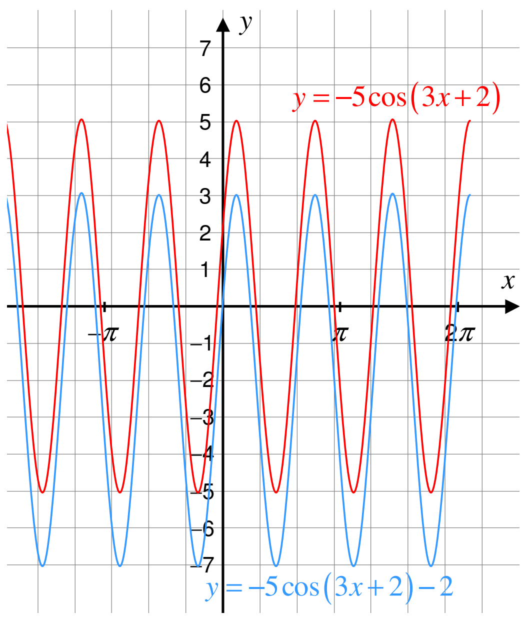


Figure 9.26

Example 6 Graph $f(x) = \frac{1}{2}\cos\left(\frac{\pi}{2}x + \frac{\pi}{2}\right)$ for one cycle.

Solution As $\frac{\pi}{2}x+\frac{\pi}{2}$ varies from 0 to 2π, x-varies from –1 to 3.

The graph completes one full cycle on the interval [–1, 3]

x = –1, 0, 1, 2, 3 divides [–1, 3] into four equal parts.

Using the following table, sketch the graph for one cycle.

x	–1	0	1	2	3
$\frac{1}{2}\cos\left(\frac{\pi}{2}x+\frac{\pi}{2}\right)$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$

Step 4, 5

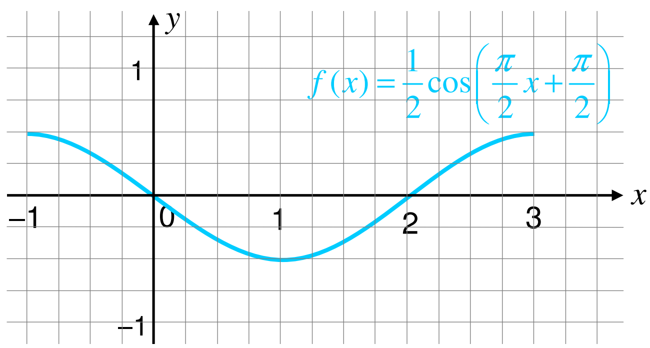


Figure 9.27

Exercise 9.4

Draw the graphs of each of the following trigonometric functions for one cycle. Indicate the amplitude, period, and phase shift.

1 $f(x)=-\frac{1}{2}\sin(2x-1)$

2 $f(x)=\frac{1}{2}\cos(3x+2)$

3 $f(x)=3\sin\left(\frac{1}{2}x+3\right)-2$

4 $f(x)=\sin(\pi x)+3$

5 $f(x)=2\cos(2x-\pi)$

6 $f(x)=3-2\cos\left(\frac{x}{2}\right)$

7 $f(x)=-\frac{3}{2}\sin\left(3x+\frac{3\pi}{4}\right)$

8 $f(x)=2-\frac{1}{2}\cos\left(\frac{3\pi}{2}x+\frac{\pi}{4}\right)$

9.3.3 Applications of Graphs in Solving Trigonometric Equations

General solutions of trigonometric equations

If you draw the graph of $y = \sin x$ and the line $y = \frac{1}{2}$ in the same coordinate system and for $0 \le x \le 2\pi$, they meet at two particular points, $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

But you know that the line $y = \frac{1}{2}$ crosses the graph of $f(x) = \sin x$ infinitely many times as shown in the figure below.

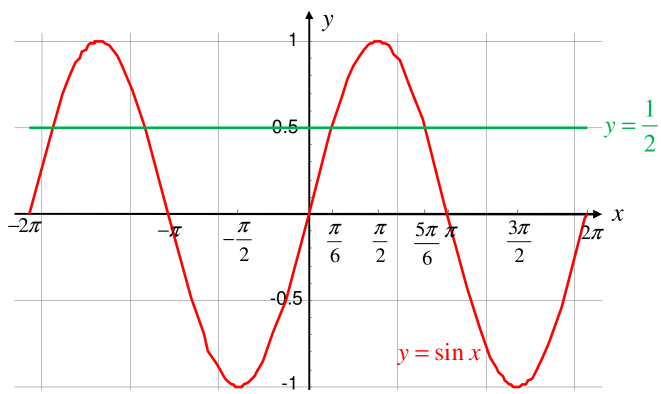


Figure 9.28

In this section, you will determine all those infinite points in terms of the particular points, the period 2π of the sine function and an integer n.

ACTIVITY 9.4

1 Draw the graphs of $f(x) = \tan x$ and the line $y = 1$ using the same coordinate system. Using the graphs

a determine the particular solution in the range $\frac{-\pi}{2} < x < \frac{\pi}{2}$ that satisfies the equation $\tan x = 1$.

b find the general solution of the equation $\tan x = 1$.

c if x_1 is a particular solution of the equation $\tan x = t$ in the range $-\frac{\pi}{2} < x < \frac{\pi}{2}$, determine the general solution in terms of x_1 and π.

2 Draw the graphs of $y = \frac{1}{2}$ and $y = \cos x$ using the same coordinate system. Determine a particular solution of the equation $\cos x = \frac{1}{2}$ in the range $-\pi \le x \le \pi$.

3 Determine the general solution of $\cos x = \frac{1}{2}$ using the particular solutions, n and 2π.

From Activity 9.4 it is clear that the general solution for a trigonometric equation is expressed in terms of the particular solutions, the period and n. The following are the techniques of finding the general solution of some trigonometric equations.

I $\tan x = t;\ t \in \mathbb{R}$.

The period of tangent function is π.

If x_1 is the particular solution in the range $\frac{-\pi}{2} < x < \frac{\pi}{2}$, then the general solution set is $\{x_1 + n\pi\}$.

Example 7 Solve $\tan x = -\frac{1}{\sqrt{3}}$.

Solution: $x_1 = -\frac{\pi}{6} \Rightarrow S.S. = \left\{-\frac{\pi}{6} + n\pi\right\}$

II $\cos x = b;\ |b| \le 1$. If x_1 is a particular solution in the range $-\pi \le x \le \pi$, then $-x_1$ is a particular solution in the same range.

$\Rightarrow$ S.S $= \{2n\pi \pm x_1\}$.

Example 8 Solve $\cos x = -\frac{\sqrt{3}}{2}$.

Solution: $x_1 = \frac{5\pi}{6} \Rightarrow S.S. = \left\{2n\pi \pm \frac{5\pi}{6}\right\}$

III $\sin x = b,\ |b| \le 1$

If $b = 0$, then $\sin x = 0 \Rightarrow S.S. = \{n\pi\}$,

$\sin x = 1 \Rightarrow$ S.S $= \left\{\frac{\pi}{2} + 2n\pi\right\}$

$\sin x = -1 \Rightarrow$ S.S $= \left\{-\frac{\pi}{2} + 2n\pi\right\}$

Suppose $0 < |b| < 1$. As it is done in the activity, the line $y = b$ crosses the graph of $y = \sin x$ at exactly two points in the interval $[0, 2\pi]$.

If x_1 and x_2 are the particular solutions, then the general solution set is

$$\{x_1 + 2n\pi, x_2 + 2n\pi\}.$$

Example 9 Solve $\sin x = \frac{\sqrt{2}}{2}$.

Solution: You know that $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$.

$$\Rightarrow S.S. = \left\{\frac{\pi}{4} + 2n\pi, \frac{3\pi}{4} + 2n\pi\right\}.$$

✍Note:

$\sin x_1 = \sin(\pi - x_1) \Rightarrow x_2 = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$. Also, if x_1 is a particular solution in the interval $[0, 2\pi]$, then the general solution set of the equation $\sin x = b$, $|b| \le 1$ is $\{(-1)^n x_1 + n\pi\}$.

Example 10 Solve $\sin(4x) = -\frac{1}{2}$.

Solution: Notice that the line $y = -\frac{1}{2}$ crosses the graph of $y = \sin(4x)$ twice in the interval $\left[0, \frac{\pi}{2}\right]$.

$$\sin(4x) = -\frac{1}{2} \Rightarrow \sin(-4x) = \frac{1}{2} \Rightarrow -4x_1 = \frac{\pi}{6}, -4x_2 = \frac{5\pi}{6}$$

$$\Rightarrow x_1 = -\frac{\pi}{24}, x_2 = -\frac{5\pi}{24}$$

Thus, the particular solutions in the interval $\left[0, \frac{\pi}{2}\right]$ are

$$-\frac{\pi}{24} + \frac{\pi}{2} = \frac{11\pi}{24}, -\frac{5\pi}{24} + \frac{\pi}{2} = \frac{7\pi}{24}$$

$$\Rightarrow \text{S.S} = \left\{\frac{11\pi}{24} + \frac{n\pi}{2}, \frac{7\pi}{24} + \frac{n\pi}{2}\right\}$$

Exercise 9.5

1 Find the general solution set for each of the following trigonometric equations.

a $\sin x = -\frac{1}{2}$ b $\cos x = \frac{\sqrt{3}}{2}$ c $\tan x = \sqrt{3}$

d $2\cos^2 x + 3\sin x = 0$ e $\cos 2x + \sin^2 x = 0$ f $\sin(6x) = \frac{\sqrt{3}}{2}$

2 Solve $\sin^2 x - \sin x \cos x = 0$ over $[0, 2\pi]$

3 Find the general solution sets for each of the following trigonometric equations on the given intervals.

a $\cos x = \frac{\sqrt{3}}{2}$ and $\tan x = -\frac{\sqrt{3}}{3}$ on $[0, 2\pi]$.

b $\cos\left(\frac{\pi}{3}x - 2\right) = \frac{1}{2}$ on $[-6\pi, 6\pi]$.

c $\sec\left(\frac{3}{2}x - \frac{\pi}{3}\right) = 2$ and $\cot x < 0$ on $[0, 2\pi]$.

d $2\sin^2 x + \cos^2 x - 1 = 0$ on $[0, 2\pi]$.

9.4 APPLICATION OF TRIGONOMETRIC FUNCTIONS

In this topic, you study some of the applications of trigonometric functions to geometry, science, navigation, wave motions and optics. The laws of sines, and cosines, the double angle and half angle formulas are included in this topic.

Many applied problems can be solved by using right-angle triangle trigonometry. You will see a number of illustrations of this fact in this section.

9.4.1 Solving Triangles

In the applications of trigonometry that you consider in this section, it is necessary to find all sides and angles of a right-angled triangle. To solve a triangle means to find the lengths of all its sides and the measures of all its angles. First solve a right-angled triangle.

Example 1 Solve the right-angled triangle shown below for all unknown sides and angles.

Solution Because C = 90°, it follows that A + B = 90°, and B = 55.8°.

To solve for a, use the fact that

$$\tan A = \frac{opp}{adj} = \frac{a}{b} \text{ which implies}$$

$$a = b \tan A$$

So, $a = 19.4\times \tan 34.2^{o} \approx 13.18$.

Similarly, to solve for c, use the fact that

$$\cos A = \frac{adj}{hyp} = \frac{b}{c} \text{ which implies}$$

$$c = \frac{b}{\cos A} = \frac{19.4}{\cos 34.2^{o}} \approx 23.46$$

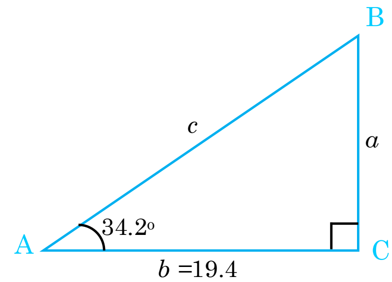


Figure 9.29

In many situations, trigonometric functions can be used to determine a distance that is difficult to measure directly. Two such cases are illustrated below.

a

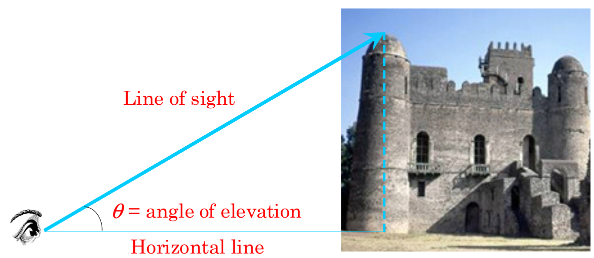


Figure 9.30

b

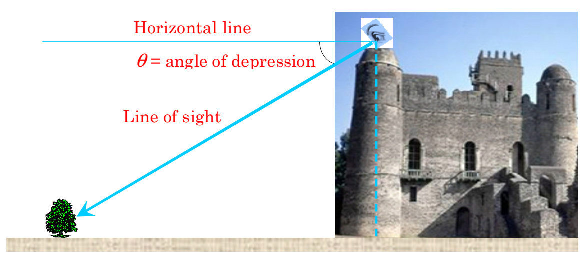


Figure 9.31

Each angle is formed by two lines: a horizontal line and a line of sight. If the angle is measured upward from the horizontal, as in a, then the angle is called an **angle of elevation**. If it is measured downward as in b, it is called an **angle of depression**.

Example 2 A surveyor is standing 50 m from the base of a large tree, as shown below. The surveyor measures the angle of elevation to the top of the tree as 15°. How tall is the tree if the surveyor is 1.72 m tall?

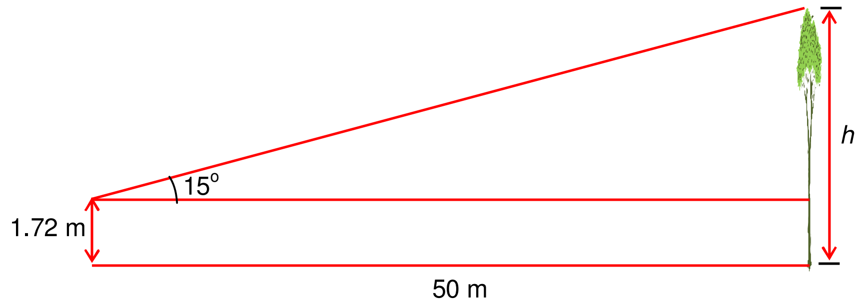


Figure 9.32

Solution The information given suggests the use of the tangent function.

Let the height of the tree be h metres. Then,

$$\tan 15^o = \frac{(h - 1.72)}{50}$$

$$0.268 \approx \frac{(h - 1.72)}{50}$$

$\Rightarrow h = (50\ (0.2679) + 1.72)$ m

$\Rightarrow h = 15.115$ m

Thus, the tree is about 15 m tall.

Example 3 A woman standing on top of a cliff spots a boat in the sea, as given in Figure 9.33. If the top of the cliff is 70 m above the water level, her eye level is 1.6 m above the top of the cliff and if the angle of depression is 30°, how far is the boat from a point at sea level that is directly below the observer?

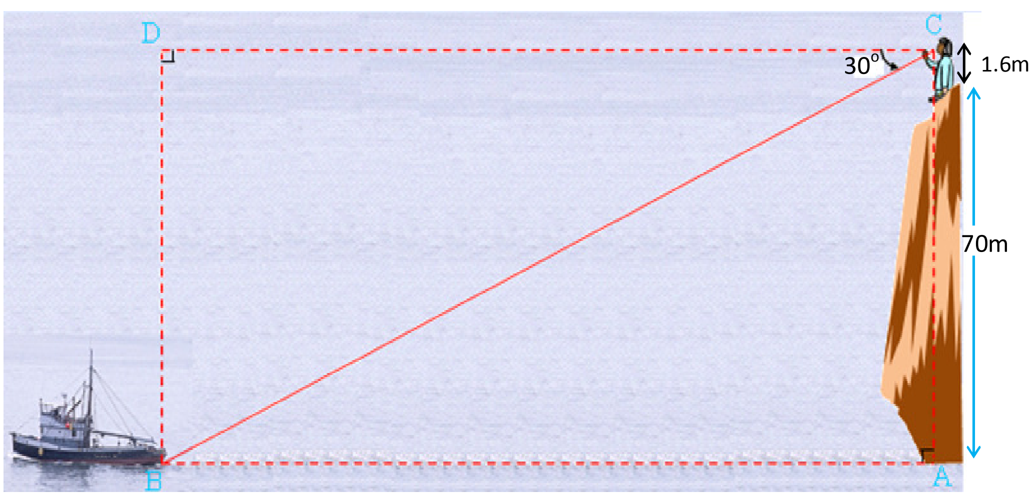


Figure 9.33

Solution In the figure, the observer's eyes are 71.6 m above the water level. Using triangle BCD, compute

$$\tan 30^\circ = \frac{BD}{DC} \Rightarrow \frac{1}{\sqrt{3}} = \frac{71.6\text{m}}{DC}$$

$$\Rightarrow DC = 71.6\sqrt{3}\text{ m}$$

$\therefore$ The boat is $71.6\sqrt{3}$ m far away from the bottom of the cliff.

Example 4 In order to measure the height of a hill, a surveyor takes two sightings from a transit 1.5 m high. The sightings are taken 1000m apart from the same ground elevation. The first measured angle of elevation is 51°, and the second is 29°. To the nearest metre, what is the height of the hill (above ground level)?

Solution First draw the figure and label the known parts. (*See* Figure 9.34)

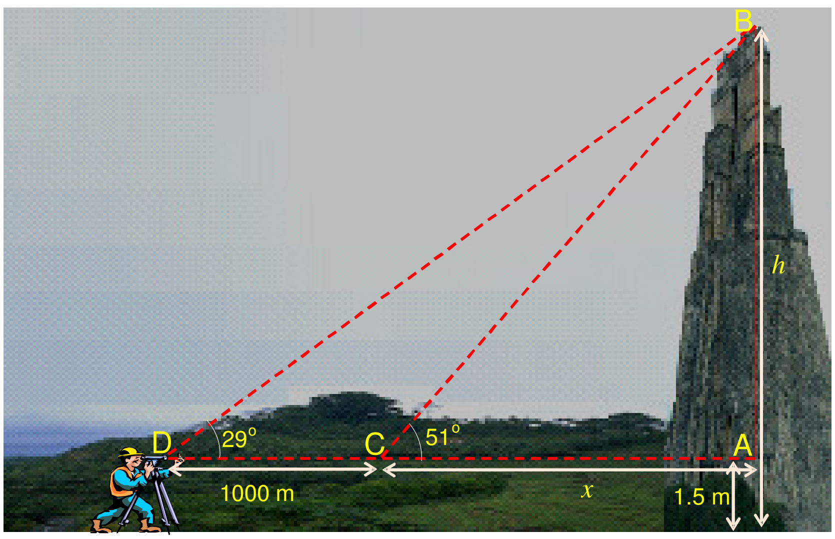


Figure 9.34

The height of the hill is $AB + 1.5$ m $= h$

But, $\tan 51^\circ = \frac{AB}{x}$ and $\tan 29^\circ = \frac{AB}{x+1000}$

$$AB = x\tan 51^\circ \text{ and } AB = (x+1000)\tan 29^\circ$$

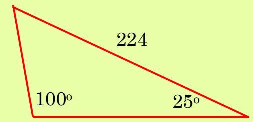
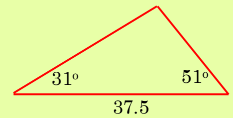
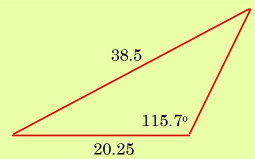
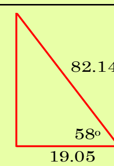
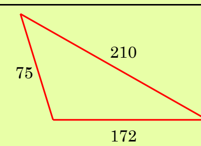
$$AB = 1.235x \text{ and } AB = (x+1000)(0.5543) = (0.5543x + 554.3)$$

Equating the two expressions for $\overline{AB}$, you have

$$1.235x = 0.5543x + 554.3 \Rightarrow x \approx 814.31$$

Thus, $AB = 1.235 \times 814.31 \approx 1005.67$ and hence $h = AB + 1.5 \text{ m} \approx 1007$ m.

The trigonometric functions can also be used to solve triangles that are not right-angled triangles. Such triangles are called oblique triangles. Any triangle, right or oblique, can be solved if at least one side and any other two measures are known. The following are the different possible conditions.

1	**AAS**: Two angles of a triangle and a side opposite to one of them are known.	**a** 
2	**ASA**: Two angles of a triangle and the included side are known.	**b** 
3	**SSA**: Two sides of a triangle and an angle opposite to one of them are known (In this case, there may be no solution, one solution, or two solutions. The latter is known as the ambiguous case)	**c** 
4	**SAS:** Two sides of a triangle and the included angle are known.	**d** 
5	**SSS:** All three sides of the triangle are known.	**e**  Figure 9.35

In order to solve oblique triangles, you need the **law of sines** and the **law of cosines**. The law of sines applies to the first three situations listed above. The law of cosines applies to the last two situations.

The law of sines

In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

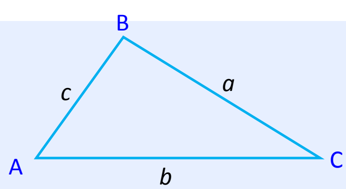


Figure 9.36

✍Note:

In any triangle, the sides are proportional to the sine of the opposite angles.

Example 5 In ΔEFG, $FG = 4.56$, $m(\angle E) = 43^\circ$, and $m(\angle G) = 57^\circ$. Solve the triangle.

Solution First draw the triangle and label the known parts. You know three of the six measures.

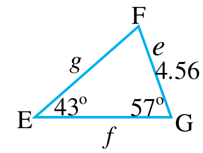


Figure 9.37

$\angle E = 43^\circ$ $\quad e = 4.56$ $\quad \angle G = 57^\circ$

$\angle F = ?$ $\quad f = ?$ $\quad g = ?$

From the figure, you have the AAS situation.

You begin by finding $m(\angle F)$.

$$m(\angle F) = 180^\circ - (43^\circ + 57^\circ) = 80^\circ$$

You can now find the other two sides, using the law of sines:

$$\frac{f}{\sin F} = \frac{e}{\sin E} \Rightarrow \frac{f}{\sin 80^\circ} = \frac{4.56}{\sin 43^\circ}$$

$$\Rightarrow f \cong 6.58$$

Also, $\dfrac{g}{\sin G} = \dfrac{e}{\sin E} \Rightarrow \dfrac{g}{\sin 57^\circ} = \dfrac{4.56}{\sin 43^\circ}$

$$\Rightarrow g \cong 5.61$$

Thus, you have solved the triangle:

$\angle E = 43^\circ$ $\quad e = 4.56,$

$\angle F = 80^\circ$ $\quad f \cong 6.58$

$\angle G = 57^\circ$ $\quad g = 5.61$

Example 6 In ΔQRS, $q = 15$, $r = 28$ and $\angle Q = 43.6^\circ$. Solve the triangle.

Solution Draw the triangle and list the known measures:

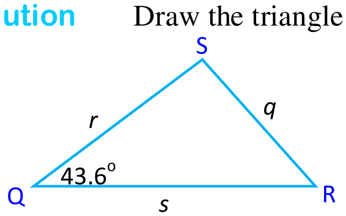


Figure 9.38

$\angle Q = 43.6^\circ$ $\quad q = 15$

$\angle R = ?$ $\quad r = 28$

$\angle S = ?$ $\quad s = ?$

You have the SSA situation and use the law of sines to find *R*:

$$\frac{q}{\sin Q} = \frac{r}{\sin R} \Rightarrow \frac{15}{\sin 43.6^\circ} = \frac{28}{\sin R}$$

$$\Rightarrow \sin R \cong 1.2873.$$

Since there is no angle with a sine greater than 1, there is no solution.

The law of cosines

In any triangle *ABC*,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

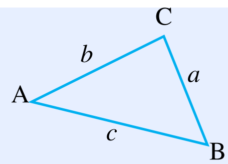


Figure 9.39

☑ Remark:

When the included angle is 90°, the law of cosines is reduced to the Pythagorean Theorem.

Example 7 Solve ΔABC, if $a = 32$, $c = 48$ and $\angle B = 125.2^\circ$

Solution You first label a triangle with the known and unknown measures.

$\angle A$ =?	$a = 32$
$\angle B = 125.2^\circ$	b =?
$\angle C$ =?	$c = 48$

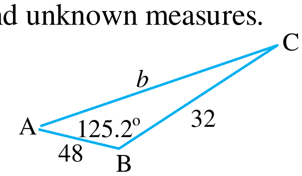


Figure 9.40

You can find the third side using the law of cosines, as follows:

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$\Rightarrow b^2 = 32^2 + 48^2 - 2(32)(48) \cos 125.2^\circ$$

$$\Rightarrow b^2 \cong 5089.8$$

$$\Rightarrow b \cong 71.34$$

You now have $a = 32$, $b \cong 71.34$ and $c = 48$, and you need to find the measures of the other two angles. At this point, you can find them in two ways, either using the law of sines or the law of cosines. The advantage of using the law of cosines is that if you solve for the cosine and find that its value is negative, then you know that the angle is obtuse. If the value of the cosine is positive, then the angle is acute. Thus you use the law of cosines:

To find angle *A*, you use

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$32^2 = (71.34)^2 + 48^2 - 2(71.34)(48) \cos A$$

$$\angle A \approx 21.55^\circ$$

The third is now easy to find:

$$\angle C \approx 180^\circ - (125.2^\circ + 21.55^\circ) \approx 33.25^\circ$$

9.4.2 Trigonometric Formulae for the Sum and Differences

In grade 10, you have seen the fundamental trigonometric identities for a single variable. In this topic, you have trigonometric identities involving the sum or difference of two variables.

For example, using your knowledge of the trigonometric values of 30° and 45°, you will then be able to determine the trigonometric values of $30^{\circ} + 45^{\circ} = 75^{\circ}$ and $45^{\circ} - 30^{\circ} = 15^{\circ}$.

Theorem 9.1 Sum and Difference Formulae

1 Sine of the Sum and the Difference

- ✓ $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- ✓ $\sin(x - y) = \sin x \cos y - \cos x \sin y$

2 Cosine of the Sum and Difference

- ✓ $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- ✓ $\cos(x - y) = \cos x \cos y + \sin x \sin y$

3 Tangent of the Sum and Difference

- ✓ $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- ✓ $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

Example 8 Find the exact values of $\sin 75^{\circ}$ and $\sin 15^{\circ}$.

Solution $\sin 75^{\circ} = \sin(30^{\circ} + 45^{\circ}) = \sin 30^{\circ}\cos 45^{\circ} + \cos 30^{\circ}\sin 45^{\circ}$

$$= \frac{1}{2}\times\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$\sin 15^{\circ} = \sin\left(45^{\circ} - 30^{\circ}\right) = \sin 45^{\circ}\cos 30^{\circ} - \cos 45^{\circ}\sin 30^{\circ}$$

$$= \frac{\sqrt{2}}{2}\times\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\times\frac{1}{2} = \frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)$$

Example 9 Find the exact value of $\cos 105^{\circ}$.

Solution $\cos 105^{\circ} = \cos(60^{\circ} + 45^{\circ}) = \cos 60 \cos 45^{\circ} - \sin 60 \sin 45^{\circ}$

$$= \frac{1}{2}\times\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}\left(1-\sqrt{3}\right)$$

Example 10 Find the exact values of

a $\tan 150^\circ$ **b** $\tan 195^\circ$

Solution

a $\tan 150^\circ = \tan(180^\circ - 30^\circ)$

$$= \frac{\tan 180^\circ - \tan 30^\circ}{1 + \tan 180^\circ \times \tan 30^\circ} = \frac{0 - \frac{1}{\sqrt{3}}}{1 + 0 \times \frac{1}{\sqrt{3}}} = -\frac{1}{\sqrt{3}}$$

b $\tan 195^\circ = \tan(150^\circ + 45^\circ) = \dfrac{\tan 150^\circ + \tan 45^\circ}{1 - \tan 150^\circ \times \tan 45^\circ}$

$$= \frac{-\frac{1}{\sqrt{3}} + 1}{1 - \left(-\frac{1}{\sqrt{3}}\right) \times 1} = 2 - \sqrt{3}$$

Theorem 9.2 Double Angle and Half Angle Formulas

1 Double Angle Formula.

✓ $\sin(2x) = 2\sin x \cos x$

✓ $\cos(2x) = \cos^2 x - \sin^2 x$

✓ $\tan(2x) = \dfrac{2\tan x}{1 - \tan^2 x}$

2 Half Angle Formula

✓ $\cos^2\left(\frac{x}{2}\right) = \dfrac{1 + \cos x}{2}$; $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\dfrac{1 + \cos x}{2}}$

✓ $\sin^2\left(\frac{x}{2}\right) = \dfrac{1 - \cos x}{2}$; $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\dfrac{1 - \cos x}{2}}$

✓ $\tan^2\left(\frac{x}{2}\right) = \dfrac{1 - \cos x}{1 + \cos x}$ for $\cos x \neq -1$;

$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

The sign is determined by the quadrant that contains $\dfrac{x}{2}$.

✍Note:

i $\cos(2x) = \cos^2 x - \sin^2 x$
$= \cos^2 x - (1 - \cos^2 x)$
giving $\cos(2x) = 2\cos^2 x - 1$

ii $\cos(2x) = \cos^2 x - \sin^2 x$
$= (1 - \sin^2 x) - \sin^2 x$
giving $\cos(2x) = 1 - 2\sin^2 x$

Example11 Find the exact values of

a $\sin\frac{\pi}{8}$ **b** $\cos 15^{o}$ **c** $\tan\frac{\pi}{8}$

Solution

a $\sin^2\frac{\pi}{8} = \frac{1-\cos\frac{\pi}{4}}{2} = \frac{1-\frac{\sqrt{2}}{2}}{2} = \frac{2-\sqrt{2}}{4}$

$\Rightarrow \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$ since $\sin\frac{\pi}{8} > 0$

b $\cos^2 15^{o} = \frac{1+\cos 30^{o}}{2} = \frac{2+\sqrt{3}}{4} \Rightarrow \cos 15^{o} = \frac{\sqrt{2+\sqrt{3}}}{2}$

c $\frac{\pi}{4} = \frac{\pi}{8} + \frac{\pi}{8} \Rightarrow \tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

$\Rightarrow 1 = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}} \Rightarrow \tan^2\frac{\pi}{8} + 2\tan\frac{\pi}{8} - 1 = 0$

Solving the quadratic equation gives $\tan\frac{\pi}{8} = -1 \pm \sqrt{2}$.

$\Rightarrow \tan\frac{\pi}{8} = \sqrt{2} - 1$, because $\tan\frac{\pi}{8} > 0$.

9.4.3 Navigation

In navigation, directions to and from a reference point are often given in terms of bearings. A bearing is an acute angle between a line of travel or line of sight and the north-south line. Bearings are usually given angles in degrees such as east or west of north, so that N θE is read as θ east of north, and so on.

Example 12 The two bearings in Figure 9.41 below are respectively,

a N30°E **b** S10°E.

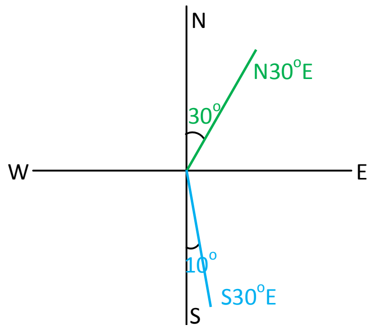


Figure 9.41

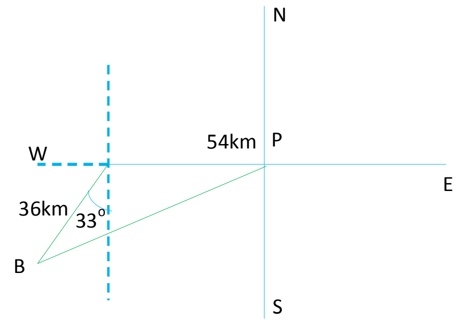


Figure 9.42

Example 13 A ship leaves a port and travels 54 km due west. It then changes course and sails 36 km on a bearing S33°W. How far is it from the port at this point? (*See* Figure 9.42)

Solution The ship is at point B. You must calculate the distance $\overline{PB}$. Using the law of cosines,

$$(\overline{PB})^2 = 54^2 + 36^2 - 2\times54\times36\times\cos 123^\circ = 2916 + 1296 - 3888 \times (-0.5446)$$

$$= 6329.4048$$

$\Rightarrow$ $PB = 79.5576$

$\Rightarrow$ The ship is about 80km from the port.

9.4.4 Optics Problem

Snell's law of refraction, which was discovered by Dutch physicist Willebrord Snell (1591 – 1626), states that a light ray is refracted (bent) as it passes from a first medium into a second medium according to the equation:

$$\frac{\sin\alpha}{\sin\beta} = \mu$$

where α is the angle of incidence and β is the angle of refraction.

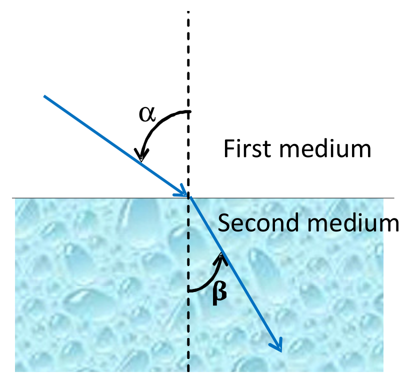


Figure 9.43

The Greek letter μ (*mu*), is called the **index of refraction** of the second medium with respect to the first.

Example 14 The index of refraction of water with respect to air is $\mu = 1.33$.
Determine the angle of **refraction**, if a ray of light passes through water with an angle of incidence $\alpha = 30^\circ$.

Solution $\mu = \dfrac{\sin\alpha}{\sin\beta} \Rightarrow 1.33 = \dfrac{\sin 30^o}{\sin\beta}$

$$\Rightarrow \sin\beta = \frac{0.5}{1.33} \approx 0.3759 \Rightarrow \beta = \sin^{-1}(0.3759)$$

$$\Rightarrow \beta = 22.1^o$$

9.4.5 Simple Harmonic Motion

The periodic nature of the trigonometric functions is useful for describing the motion of a point on an object that vibrates, oscillates, rotates, or is moved by wave motion.

In Physics, Biology, and Economics, many quantities are periodic. Examples include the vibration or oscillation of a pendulum or a spring, periodic fluctuations in the population of a species, and periodic fluctuations in a business cycle. Many of these quantities can be described by harmonic functions.

Definition 9.5

A **harmonic function** is a function that can be written in the form

$$g(t) = a\cos\omega t + b\sin\omega t. \qquad 1$$

Note that 1 can be written in the forms

$$a\cos\omega t + b\sin\omega t = A\cos(\omega t - \delta) \qquad 2$$

$$a\cos\omega t + b\sin\omega t = A\sin(\omega t + \phi) \qquad 3$$

Where $A = \sqrt{a^2+b^2}$, $(\cos\delta, \sin\delta) = \left(\frac{a}{A}, \frac{b}{A}\right)$, and $(\cos\phi, \sin\phi) = \left(\frac{b}{A}, \frac{a}{A}\right)$

In 2 or 3, the period is $\frac{2}{\omega}\pi$. The frequency f of the function is the number of complete periods per unit time. Since $y = A\cos(\omega t - \delta)$ or $y = A\sin(\omega t + \delta)$ returns to the same y value in one period equal to $\frac{2}{\omega}\pi$ time units, you have:

Natural frequency of a function

$$f = \frac{\omega}{2\pi}$$

Units of frequency are cycles/sec (also called Hertz).

Example 15 *A simple electric circuit*

In an electric circuit, such as the one in the figure on the right, an electromotive force (EMF) E (volts), usually a battery or generator, drives an electric charge Q (coulombs) and produces a current I (amperes). In the circuit shown in Figure9.44, a resistor of resistance R (Ohms) is a component of the circuit that opposes the current, dissipating the energy in the form of heat. It produces a drop in the voltage given by Ohm's law:

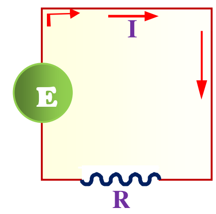


Figure 9.44

$$E = RI$$

The electromotive force (EMF) may be direct or alternating. A direct EMF is given by a constant voltage. An alternating EMF is usually given as a sine function:

$$E = E_o \sin \omega_o t,\ E_o > 0$$

Since $-1 \le \sin \omega_o t \le 1$, you see that

$$-E_o \le E \le E_o$$

Thus, E_o is the maximum voltage, and $-E_o$ is the minimum voltage.

Example 16 Suppose that an EMF of $E = 10 \sin \frac{\pi}{4} t$ volts is connected in the circuit of Figure 9.45 above with a resistance of 5 ohms.

a What is the period of the EMF?

b What is the frequency?

c What is the maximum current in the system?

Solution

a $\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{4}} = \frac{8\pi}{\pi} = 8$

b $\text{frequency} = \frac{\omega}{2\pi} = \frac{1}{8}$ cycles/sec

c from the equation E = RI, we have:

$$I = \frac{E}{R} = \frac{10 \sin \frac{\pi}{4} t}{5} = 2 \sin \frac{\pi}{4} t \text{ ampere.}$$

The maximum current is 2 amperes.

Example 17 Given the equation for simple harmonic motion $d = 6 \cos \frac{3}{4}\pi t$, find

a the maximum displacement

b the frequency

c the value of d when $t = 4$

d the least positive value of t for which $d = 0$.

Solution

a The maximum displacement is 6, because the maximum displacement from the point of equilibrium is the amplitude.

b Frequency $= \frac{\omega}{2\pi} = \frac{\frac{3}{4}\pi}{2\pi} = \frac{3}{8}$ cycle/unit time.

c $d = 6 \cos \left(\frac{3\pi}{4}(4)\right) = 6 \cos 3\pi = 6(-1) = -6$

d To find the least positive value of t for which $d = 0$, solve the equation

$d = 6 \cos \frac{3}{4}t = 0$ to obtain

$\frac{3}{4}\pi t = \frac{\pi}{2}, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$ which implies $t = \frac{2}{3}, 2, \frac{10}{3}, \ldots$

Thus, the least positive value of t is $t = \frac{2}{3}$

Vibrations, such as those created by plucking a violin string or striking a wooden tube, cause sound waves, which may or may not be audible to the human ear. Often, sound waves are **sinusoidal** and can therefore be written in the form

$$y = a \sin \omega t$$

Here you assume that there is no phase shift [i.e. $\phi = 0$ in equation 3]. The amplitude a is related to the loudness of the sound, which is measured in decibels.

Example 18 Niddle C is struck on a piano with amplitude of $a = 2$. The frequency of niddle C is 264 cycles/sec. Write an equation for the resulting sound wave.

Solution With $a = 2$, we have

$$y = 2 \sin \omega t$$

But, frequency $= \frac{\omega}{2\pi} = 264$

So $\omega = 264\,(2\pi) = 528\pi$. Thus, $y = 2 \sin 528\pi t$ is the equation of the sound wave.

Exercise 9.6

1. A flying airplane is sighted in a line from two observation stations *A* and *B*. The angle of elevation of the airplane from *A* is 30° and from *B* is 60°. *A* and *B* are on the same side of the airplane. If the distance between *A* and *B* is 1200 m, find the altitude of the airplane.

2. Solve each of the following triangles. Approximate the answers to two decimal places.

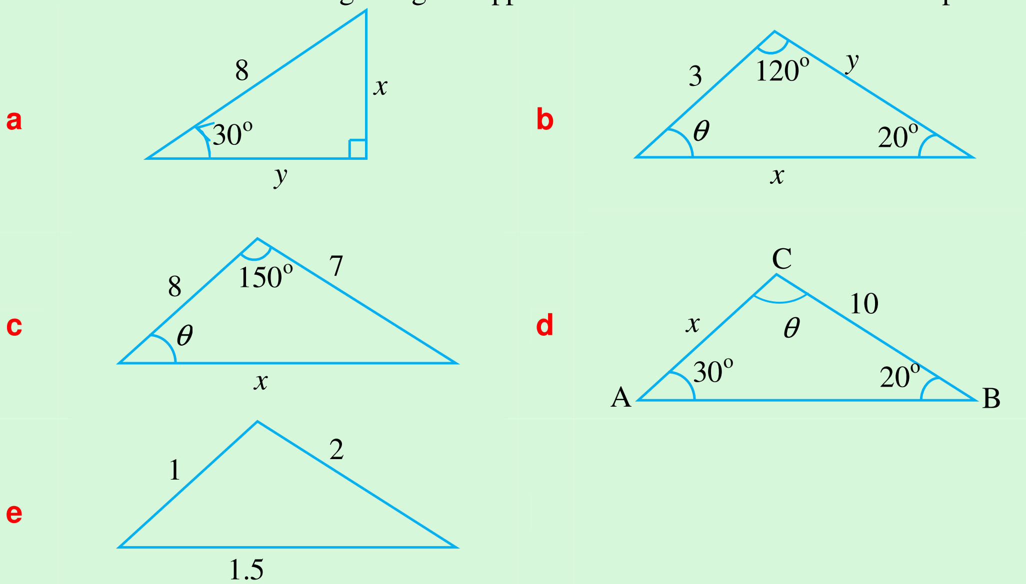


Figure 9.45

3. The angle of elevation of the top of a building is found to be 70° as measured from a point on a level ground. If the angle of elevation of a point on the building that is 3 m below the top is 60° as measured from the same point on the ground, find the height of the building.

4. Given below is an isosceles trapezium with shorter base b units and the congruent sides a units long. If the base angle measures θ°, express the area of the trapezium in terms of a, b, $\sin\theta$ and $\cos\theta$.

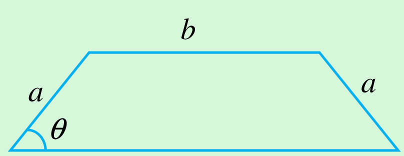


Figure 9.46

5 Two boats A and B leave the same port P at the same time. A travels 60 km in the direction N 75^o W to port Q and B travels 80 km in the direction S 45^o W to port R. Find the distance between port Q and port R.

6 The refraction index of water with respect to air is $\mu = 1.33$. Determine the angle of refraction β of a ray of light that strikes the water body with an angle of incidence $\alpha = 45^o$.

7 Find the exact values of the following trigonometric functions without using a calculator or tables.

a $\sin 165^o$ **b** $\cos 105^o$ **c** $\tan \frac{17}{12}\pi$

d $\sec\frac{11}{12}\pi$ **e** $\cot \frac{19}{12}\pi$ **f** $\csc\frac{13}{12}\pi$

8 Simplify each of the following expressions.

a $\dfrac{\tan 175^o - \tan 130^o}{1+\tan 175^o \times \tan 130^o}$ **b** $\dfrac{\sin x + \tan x}{\csc x + \cot x}\times \cot x$

c $\dfrac{\sin(2x)+\sin(4x)}{\cos(2x)-\cos(4x)}$ **d** $\dfrac{\cot x}{1-\tan x}+\dfrac{\tan x}{1-\cot x}-\dfrac{2}{\sin(2x)}$

e $\sin\left(\sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{5}{13}\right)\right)$

9 An alternating current generator generates a current given by the formula I = 20 sin $40\pi t$, where t is time in seconds.

a Determine the amplitude and the period.

b What is the frequency of the current?

10 An aeroplane is flying in a direction S 15^o E at an air speed of 1403 km/hr. A steady wind of 56 km/hr is blowing in the direction of S 30^o W. Find the velocity of the aeroplane relative to the ground.

11 A boat directed N 75^oE is crossing a river at a speed of 20 km/hr relative to the water. The river is flowing in the direction of S 30^o E at 6 km/hr. Find the velocity of the boat relative to the ground.

12 In ΔXYZ, $x = 23.5$, $y = 9.8$, $\angle X = 39.7^o$. Solve the triangle.

13 In ΔABC, $b = 15$, $c = 20$, and $\angle B = 29^o$. Solve the triangle.

14 If $x = a\cos\theta - b\sin\theta$ and $y = a\sin\theta + b\cos\theta$, express $x^2 + y^2$ in terms of a and b

15 **Simple pendulum:** An object consisting of a point mass m is suspended by a weightless string of length ℓ as shown in Figure 9.47. If it is pulled to one side of its vertical position and released, it moves periodically to the right and to the left.

Let y denote the displacement of the mass from its vertical position, measured along the arc of the swing at time t. suppose that $y = a$ when $t = 0$, the instance of release. Then, if a is not too large, the quantity y will approximately oscillate according to the simple harmonic model $y = a\cos\omega t$ with period $T = 2\pi\sqrt{\frac{l}{g}}$, where g is the acceleration of gravity.

$g \approx 32$ feet/sec^2 or $g \approx 9.8$ m/sec^2

If $\ell = 1.2$ m and $a = 0.06$ m, determine the equation for y as a function of t and find

a the period.

b the angular frequency.

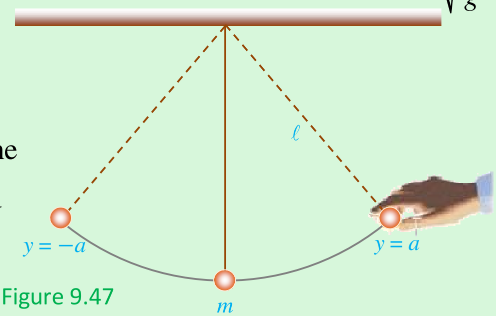


Figure 9.47

Key Terms

arccosecant	**cosecant**	**secant**
arccosine	**cosine**	**sine**
arccotangent	**cotangent**	**sinusoidal**
arcsecant	**harmonic motion**	**tangent**
arcsine	**laws of cosines**	**trigonometric identities**
arctangent	**laws of sines**	

Summary

1 ***The Reciprocal Trigonometric Functions:***

i ***The Cosecant Function:*** the reciprocal of sine function, $y = \csc x$.

✓ $\csc x = \dfrac{1}{\sin x}$

✓ Domain $= \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$

✓ Range $= (-\infty, -1] \cup [1, \infty)$

✓ Period $= 2\pi$

ii ***The Secant Function:*** the reciprocal of cosine function, $y = \sec x$.

✓ $\sec x = \dfrac{1}{\cos x}$

✓ Domain = $\mathbb{R} \setminus \left\{ (2k+1)\dfrac{\pi}{2} : k \in \mathbb{Z} \right\}$

✓ Range = $(-\infty, -1] \cup [1, \infty)$

✓ Period = 2π

iii ***The Cotangent Function:*** the reciprocal of tangent function, $y = \cot x$.

✓ $\cot x = \dfrac{1}{\tan x}$

✓ Domain = $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$

✓ Range = $\mathbb{R}$

✓ Period = π

2 Inverse Trigonometric Functions

i The Inverse Sine or Arcsine

$\sin^{-1} x = y$, if and only if $x = \sin y$ and $\dfrac{-\pi}{2} \le y \le \dfrac{\pi}{2}$.

ii The Inverse Cosine or Arccosine

$\cos^{-1} x = y$, if and only if $x = \cos y$ and $0 \le y \le \pi$

iii The Inverse Tangent or Arctangent

$\tan^{-1} x = y$, if and only if $x = \tan y$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

iv The Inverse Cosecant or Arccosecant

$\csc^{-1} x = y$, if and only if $x = \csc y$ and $\dfrac{-\pi}{2} \le y \le \dfrac{\pi}{2}$ with $y \ne 0$.

$\csc^{-1} x = \sin^{-1}\left(\dfrac{1}{x}\right); |x| \ge 1$

v The Inverse Secant or Arcsecant

$\sec^{-1} x = y$, if and only if $x = \sec y$ and $0 \le y \le \pi$ with $y \ne \dfrac{\pi}{2}$.

$\sec^{-1} x = \cos^{-1}\left(\dfrac{1}{x}\right); |x| \ge 1$

vi ***The Inverse Cotangent or Arccotangent***

$\cot^{-1} x = y$ if and only if $x = \cot y$ and $0 < y < \pi$.

$\cot^{-1} x = \dfrac{\pi}{2} - \tan^{-1} x$

3 *Graphs of some trigonometric functions.*

$y = a \sin(kx + b) + c$ and $y = a \cos(kx + b) + c$,

i **Amplitude** $= |a|$

ii **Period**, $P = \dfrac{2\pi}{k}$; $k > 0$

When $k < 0$, use the symmetric property

iii **Range** $= [c - |a|,\ c + |a|]$

iv **Phase angle** $= -b$

v **Phase shift** $= \dfrac{-b}{k}$

4 *Applications of Trigonometric Functions*

Solving a triangle

i **The Law of Sines** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

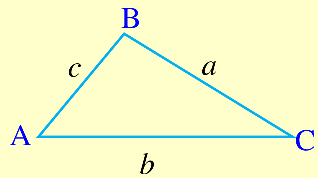


Figure 9.48

ii **The Law of Cosines**

$c^2 = a^2 + b^2 - 2ab \cos C$, $b^2 = a^2 + c^2 - 2ac \cos B$,

$a^2 = b^2 + c^2 - 2bc \cos A$.

iii Trigonometric Formulae for the sum and difference

The addition and difference identities

✓ $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

✓ $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

✓ $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

Double – Angle Formulas

✓ $\cos(2x) = \cos^2 x - \sin^2 x$

$\cos(2x) = 2\cos^2 x - 1$

✓ $\cos(2x) = 1 - 2\sin^2 x$

✓ $\sin(2x) = 2 \sin x \cos x$

✓ $\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$

Half Angle Formulas

✓ $\cos^2\left(\dfrac{x}{2}\right) = \dfrac{1+\cos x}{2}$

✓ $\sin^2\left(\dfrac{x}{2}\right) = \dfrac{1-\cos x}{2}$

✓ $\tan^2\left(\dfrac{x}{2}\right) = \dfrac{1-\cos x}{1+\cos x}$; $\cos x \neq -1$

5 *Simple Harmonic Motion*

$g(t) = a\cos(\omega t) + b\sin(\omega t)$

✓ **period**, $P = \dfrac{2\pi}{\omega}$

✓ **frequency**, $f = \dfrac{\omega}{2\pi}$

Review Exercises on Unit 9

1 Prove the following identities.

a $\cot(x+\pi) = \cot x$ **b** $\cot(-x) = -\cot x$

c $\sec(-x) = \sec x$ **d** $\csc(-x) = -\csc x$

2 Find each value.

a $\sec\dfrac{\pi}{4}$ **b** $\csc\dfrac{\pi}{6}$ **c** $\cot\dfrac{\pi}{2}$

3 Explain how the graph of $y = \csc x$ is related to the graph of $y = \sec x$.

4 Find a function f of the form $f(x) = a\sin(kx)$ satisfying the given properties

a amplitude 3 and period $\dfrac{2}{5}\pi$

b amplitude $\dfrac{2}{5}$ and $f(3) = 0$

c peak at $\left(\dfrac{\pi}{3}, 5\right)$

d amplitude 2, the graph passes through $\left(\dfrac{\pi}{3}, 0\right)$

5 Repeat problem number 4, if $f(x) = a \cos(kx)$.

6 Find each value.

a $\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right)$ **b** $\tan^{-1}(1)$ **c** $\tan^{-1}\left(-\sqrt{3}\right)$

7 Using a calculator or tables, find each value

a arcsin (0.0941) **b** arccos (0.5525) **c** arctan (–2.4147)

8 Find the exact values of each of the following without using a calculator or tables.

a $\sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$ **b** $\sin\left(\sin^{-1}(0.025)\right)$

c $\cos\left(\cos^{-1}\left(-\frac{3}{4}\right)\right)$ **d** $\sin\left(\cos^{-1}\left(\frac{1}{8}\right)\right)$

e $\cos\left(\sin^{-1}(x)\right)$ for $|x| \le 1$ **f** $\sin\left(\cos^{-1}(x)\right)$ for $|x| \le 1$

g $\tan\left(\cos^{-1}\left(\frac{4}{9}\right)\right)$ **h** $\sin\left(2 \tan^{-1}\left(-\frac{4}{5}\right)\right)$

9 If $\sin(\alpha+\theta) = \frac{55}{73}$ and $\sin\theta = \frac{3}{5}$, find sin α

10 If $\sin x = -\frac{12}{37}$, $\pi < x < \frac{3}{2}\pi$, find $\cos\left(\frac{x}{2}\right)$.

11 Draw the graph of each of the following functions for one cycle.

a $f(x) = 2\sin\left(x - \frac{\pi}{2}\right)$ **b** $f(x) = \cos\left(-\frac{1}{2}x + \frac{\pi}{4}\right)$

c $f(x) = 3 - \sin\left(\frac{1}{2}x + \frac{\pi}{4}\right)$ **d** $f(x) = 2\cos\left(\frac{\pi}{4}x\right) + 3$

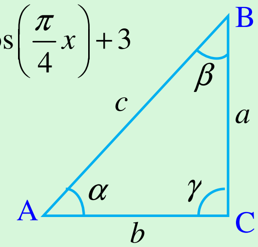


Figure 9.49

12 Use the law of sines to solve ΔABC if

a $a = 5$, $\beta = 50^\circ$, $\gamma = 70^\circ$

b $a = 5$, $b = 3$, $\alpha = 45^\circ$

c $a = 11$, $b = 24$, $\alpha = 59.5^\circ$

13 Use the law of cosines to solve ΔABC if

a $a = 5$, $b = 6$, $\gamma = 60^\circ$ **b** $b = 8$, $c = 7$, $\alpha = 30^\circ$

c $a = 20$, $c = 30$, $\beta = 110^\circ$

14 Solve each of the following trigonometric equations.

a $\sin(2x) = \sqrt{3}\ \sin x$ **b** $\sin(2x) = -\frac{1}{\sqrt{2}}$

c $\tan\left(3x - \frac{\pi}{4}\right) = \sqrt{3}$ **d** $2\sin x = \sin(2x)$

e $\tan\left(\frac{x}{2}\right) - 2\sin x = 0$

15 Two drivers A and B leave the same place at the same time. If A drives 80 km/hr in the direction of N 30°E and B drives 90km/hr in the direction of S 60°W, how far apart are they after $1\frac{1}{2}$ hours?

16 A tower 15 m high is on the bank of a river. It is observed that the angle of depression from the top of the tower to a point on the opposite shore is 30° and the angle of depression from the base of the tower to the same point on the opposite shore is observed to be 15°. Find the width of the river.

17 The refraction index of ice with respect to air is $\mu = 1.309$. Determine the angle of refraction β of a ray of light that strikes a block of ice with an angle of incidence $\alpha = 40^{\circ}$.

18 Prove each of the following trigonometric identities.

a $\cos^4 x - \sin^4 x = \cos(2x)$ **b** $\frac{\cos(2x)}{1 + \sin(2x)} = \frac{\cot x - 1}{\cot x + 1}$

19 Simplify $y = \tan(2\sin^{-1} x)$ in terms of x.

20 The population (in hundreds) of a species of bird in an area is modelled by the function

$$P(t) = 5 + 3\sin\left(\frac{2\pi t}{5}\right);\ 0 \le t \le 12.$$

where t is the time in months,

Determine:

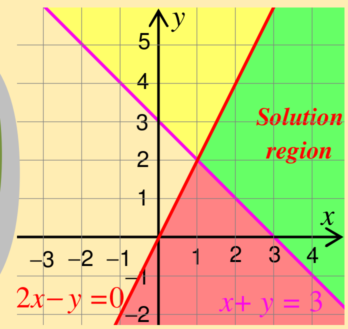
a the initial population.

b the largest and smallest populations.

c the first time in which the population reaches 350 birds.

d the population after one year.

Unit 10



INTRODUCTION TO LINEAR PROGRAMMING

Unit Outcomes:

After completing this unit, you should be able to:

- *identify regions of inequality graphs.*
- *create real life examples of linear programming problems using inequalities and solve them.*

Main Contents:

INTRODUCTION

Many real life problems involve finding the optimum (maximum or minimum) value of a function under certain conditions. In particular, linear programming is a field of mathematics that deals with the problem of finding the maximum or minimum value of a given linear function, known as the objective function, subject to certain conditions expressed as linear inequalities known as constraints. The objective function may be profit, cost, production capacity or any other measure of effectiveness, which is to be obtained in the best possible or optimal manner. The constraints may be imposed by different resource limitations such as market demand, labour time, production capacity, etc.

HISTORICAL NOTE

Leonid Vitalevich Kantorovich (1912-1986)

A Soviet Mathematician, and Economist, received his doctorate in 1930 at the age of eighteen. One of his most fundamental works on economics was The Best Use of Economic Resources (1959). Kantorovich pioneered the technique of linear programming as a tool of economic planning, having developed a linear programming model in 1939. He was a joint winner of the 1975 Nobel Prize for economics for his work on the optimal allocation of scarce resources.

OPENING PROBLEM

A man wants to fence a plot of land in the shape of a triangle whose vertices are at the points A (4, 1), B (2, 5) and C (–1, 0).

i identify this region in the *xy*-plane;

ii find the equation of the lines that pass through the sides of this region;

iii express the region bounded by the fences using in equalities.

10.1 REVISION ON LINEAR GRAPHS

Given a non horizontal line ℓ on the xy-coordinate plane, it intersects with the x-axis at exactly one point. The angle α measured from the x-axis to ℓ in the counter clockwise direction is called the **inclination** of the line $(0 \leq \alpha < 180^\circ)$.

In order to determine the equation of ℓ, we pick two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on ℓ as shown in Figure 10.1. Then we define the slope m of ℓ by

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}, \text{ for } x_1 \neq x_2.$$

Since $\tan\alpha = \dfrac{\text{Opposite side}}{\text{Adjacent side}} = \dfrac{y_2 - y_1}{x_2 - x_1}$, we have $m = \tan\alpha$

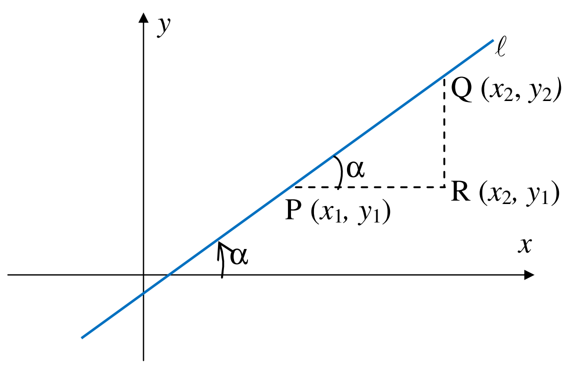


Figure 10.1

Example 1 The slope of a line ℓ passing through the points P (3, –2) and Q (–1, 3) is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-(-2)}{-1-3} = -\dfrac{5}{4}$.

Two non-vertical lines ℓ_1 and ℓ_2 with slopes m_1 and m_2, respectively, are **parallel** if and only if they have the same slope; i.e, $m_1 = m_2$.

ACTIVITY 10.1

a Find the value of k so that the line passing through the points P (1, –2) and Q (k, 3) has slope 5.

b Verify that the line ℓ_1 through the points A (1, 1) and B (–2, 3) is parallel to the line ℓ_2 through the points C (3, 2) and D (–3, 6).

An **equation** of a line ℓ is an equation in two variables x and y such that a point P (x, y) is on ℓ if and only if x and y satisfy the equation.

Recall that if a line ℓ has slope m and passes through a point P (x_1, y_1), then the point-slope form of equation of ℓ is given by

$$y - y_1 = m(x - x_1)$$

If the line passes through (0, 0), its equation is $y = mx$

Example 2 The equation of the line passing through P (–2, 3) with slope $m = 2$ is given by $y - 3 = 2(x - (-2)) = 2(x + 2) = 2x + 4$ or $y = 2x + 7$.

If the y-intercept of a line with slope m is $(0, b)$, then its equation in the slope-intercept form is

$$y = mx + b$$

Example 3 The equation of a line ℓ with slope $\frac{1}{2}$ and y-intercept –3 is given by

$$y = \frac{1}{2}x - 3 \quad \text{or} \quad 2y = x - 6$$

To sketch the graph of this line in the xy-plane, we need to plot two points. One of these is the y-intercept (0, –3). To get a second point, take $x = 2$. Then $y = -2$, so that the point (2, –2) is on the line.

Using these two points, the line ℓ can be drawn as shown in Figure 10.2. If a line ℓ has the same slope $\frac{1}{2}$ (i.e., ℓ_1 is parallel to ℓ) and has y-intercept –1, its equation is $y = \frac{1}{2}x - 1$ or $2y = x - 2$.

Its graph is shown in Figure 10.2.

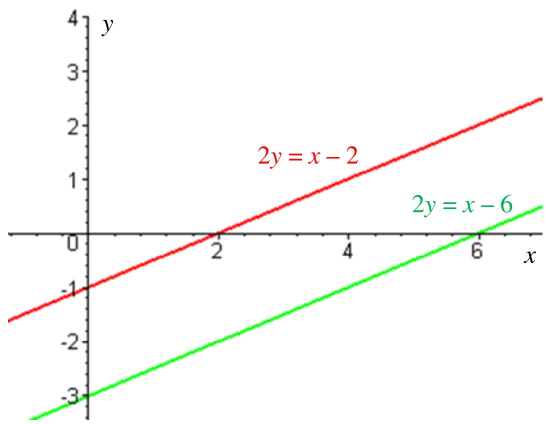


Figure 10.2

Any equation of a line can be reduced to the form $ax + by = c$ where $a, b, c \in \mathbb{R}$ with $a \neq 0$ or $b \neq 0$.

Example 4 If a line ℓ passes through P (1, –3) and Q (2, 2), then its slope is

$$m = \frac{2+3}{2-1} = 5$$

Its equation in slope-intercept form is

$$y - 2 = 5(x - 2) = 5x - 10 \quad \text{or } y = 5x - 8 \text{ (slope 5, } y\text{-intercept } (0, -8))$$

This can be written in the form $5x - y = 8$, with $a = 5$, $b = -1$ and $c = 8$.

Note:

1 An equation of a vertical line passing through the point (h, k) is given by $x = h$. A vertical line has no slope.

2 An equation of a horizontal line passing through the point (h, k) is given by $y = k$. A horizontal line has zero slope.

3 Two lines are perpendicular, if and only if their slopes are negative reciprocals of each other. That is, if ℓ_1 has slope m_1 and ℓ_2 has slope m_2, then ℓ_1 is perpendicular ℓ_2, if and only if $m_1 m_2 = -1$.

Exercise 10.1

1 Determine the equation of the line

  a that has slope 4 and passes through P (–1, 3).

  b that passes through the points P (1, 2) and Q (–4, 1).

  c whose slope is –2 with y-intercept (0, 5).

2 Determine the value of k so that the line with equation $4x + ky = 8$ is parallel to the line with equation $x + 2y = 0$.

3 Draw the graphs of the following lines using the same coordinate axes.

  a $y = 2x - 1$ b $y = 2x + 3$ c $3x - 2y = 4$

10.2 GRAPHICAL SOLUTIONS OF SYSTEMS OF LINEAR INEQUALITIES

In this section, you use graphs to determine the solution set of a system of linear inequalities in two variables.

Every line ℓ: $ax + by = c$ in the plane divides the plane into two regions, one on each side of the line. Each of these regions is called a **half plane**. A vertical line ℓ: $x = a$ divides the plane into left and right half planes. A point (x, y) is to the left of the line $x = a$, if and only if, $x < a$. Hence the graph of the inequality $x < a$ is the half plane lying to the left of the line $x = a$. Similarly, the graph of the inequality $x > a$ is the half plane lying to the right of the line $x = a$.

Example 1 Let ℓ be the vertical line $x = 2$.

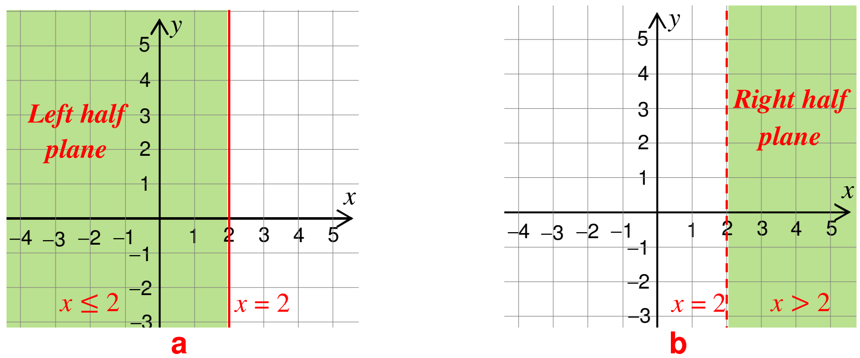


Figure 10.3

Observe that in a the left half plane $x \le 2$ contains the points on the line $x = 2$ and hence the line is a bold (unbroken) line; whereas in b the right half plane $x > 2$ does not include the points on the line $x = 2$ (broken line).

A non-vertical line divides the plane into two regions which can be called **upper and lower half planes.**

Example 2 Consider the graph of the linear equation $2x - y = 3$ and the related linear inequalities $2x - y \ge 3$ and $2x - y < 3$. First graph the line $2x - y = 3$ by plotting two points on the line. To identify which half plane belongs to which inequality, test a point that does not lie on the line (usually the origin).

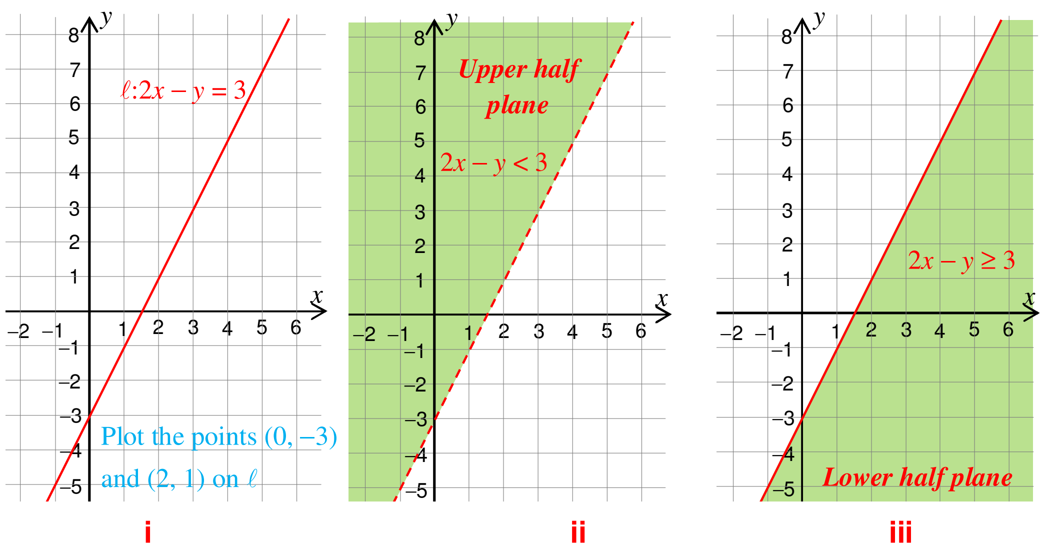


Figure 10.4

Test (0, 0); $2(0) - 0 = 0 < 3$

Observe the broken line for $2x - y < 3$ and solid line for $2x - y \ge 3$.

ACTIVITY 10.2

Draw the graph of each of the following inequalities:

a	$x \geq 0$	**b**	$y < -1$	**c**	$y \geq 3x$
d	$x > 2y$	**e**	$4x + y \geq 1$	**f**	$-x + 3y < 2$

A **system of linear inequalities** is a collection of two or more linear inequalities to be solved simultaneously. A **graphical solution** of a system of linear inequalities is the graph of all ordered pairs (x, y) that satisfy all the inequalities. Such a graph is called the **solution region** (or **feasible region**).

Example 3 Find a graphical solution to the system of linear inequalities.

$$\left.\begin{aligned} x + y &\geq 3 \\ 2x - y &\geq 0 \end{aligned}\right\}$$

Solution First draw the lines $x + y = 3$ and $2x - y = 0$ by plotting two points for each line. Then shade the regions for the two inequalities.

The solution region is the intersection of the two regions. To find the point of intersection of the two lines, solve

$$\left.\begin{aligned} x + y &= 3 \\ 2x - y &= 0 \end{aligned}\right\} \text{ simultaneously, to get the point } (1, 2).$$

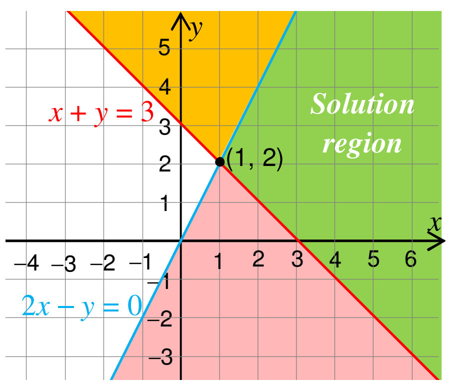


Figure 10.5

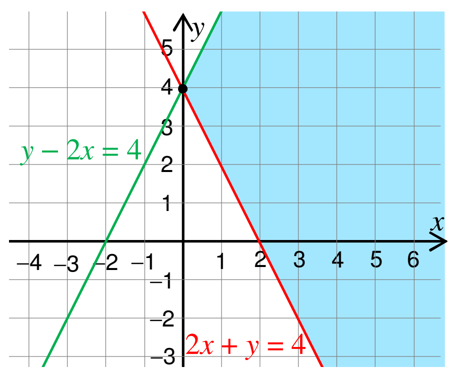


Figure 10.6

Example 4 Draw the solution region of the system of linear inequalities.

$$\left.\begin{aligned} y - 2x &\leq 4 \\ 2x + y &\geq 4 \end{aligned}\right\}$$

Solution Draw the two lines ℓ_1: $y - 2x = 4$ and ℓ_2: $2x + y = 4$ and identify their point of intersection P (0, 4). The solution region, which is the intersection of the two half planes, is shaded in Figure 10.6.

> **Definition 10.1**
>
> A point of intersection of two or more boundary lines of a solution region is called a **vertex** (or a **corner point**) of the region.

Example 5 Solve the following system of linear inequalities.

$$\left.\begin{array}{r} 2x + y \le 22 \\ x + y \le 13 \\ 2x + 5y \le 50 \\ x \ge 0 \\ y \ge 0 \end{array}\right\}$$

Solution The last two inequalities $x \ge 0$ and $y \ge 0$ are known as non-negative inequalities (or non-negative requirements). They indicate that the solution region is in the first quadrant of the plane.

Draw the lines

$$\ell_1 : 2x + y = 22,\ \ell_2 : x + y = 13 \text{ and } \ell_3 : 2x + 5y = 50$$

To determine the solution region test the point O (0, 0) which is not in any of these 3 lines, and find the intersection of all half planes to get the shaded region in Figure 10.7.

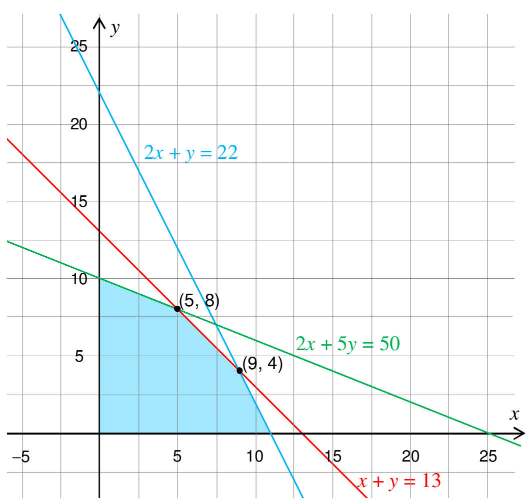


Figure 10.7

This solution region has five corner points. The vertices O (0, 0), P (0, 10) and Q (11, 0) can be easily determined. To find the other two vertices R and S solve simultaneously the following two pairs of equations:

$$\left.\begin{aligned}\ell_1 &: 2x + y = 22\\ \ell_2 &: \ \ x + y = 13\end{aligned}\right\} \text{ and } \left.\begin{aligned}\ell_2 &: \ \ x + y = 13\\ \ell_3 &: 2x + 5y = 50\end{aligned}\right\}$$

to get S (9, 4) to get R (5, 8)

Observe that the point of intersection of ℓ_1 and ℓ_3 is not a corner point of the solution region.

Definition 10.2

A solution region of a system of linear inequalities is said to be **bounded**, if it can be enclosed by a rectangle, otherwise it is **unbounded**.

Thus the solution region of Example 5 is bounded, while that of Example 4 is unbounded.

Exercise 10.2

Find a graphical solution for each of the following.

a $x \geq 0$	**b** $x - y \leq 2$	**c** $x \geq 0$	**d** $x, y \geq 0$
$y \geq 0$	$x + y \geq 2$	$y \geq 0$	$2x + 3y \leq 60$
$2x + 3y \leq 4$	$x + 2y \leq 8$	$x + y \geq 8$	$2x + y \leq 28$
	$x \leq 4$	$3x + 5y \geq 30$	$4x + y \leq 48$

10.3 MAXIMUM AND MINIMUM VALUES

Group Work 10.1

Find two positive numbers whose sum is at least 15 and whose difference is at most 7, such that their product is

a minimum **b** maximum

Many applications in business and economics involve a process called **optimization**, in which you are asked to find the maximum or minimum value of a quantity. In this section you will study an optimization strategy called **linear programming**.

Definition 10.3

Suppose f is a function with domain I = $\{ x | a \leq x \leq b\}$

i A number $M = f(c)$ for some c in I is called the **maximum** value of f on I, if

$M \geq f(x)$,for all x in I.

ii A number $m = f(d)$ for some d in I is called the **minimum** value of f on I, if

$m \leq f(x)$,for all x in I.

iii A value which is either a maximum or a minimum is called an **optimum** (or **extremum**) value of f on I.

Many optimization problems involve maximizing or minimizing a linear function **(the objective function)** subject to one or more linear equations or inequalities **(constraints)**.

In this section, problems with only two variables are going to be considered since such problems can easily be solved by a graphical method.

Example 1 Find the values of x and y which will maximize the value of the objective function

$Z = f(x, y) = 2x + 5y$, subject to the linear constraints:

$$x \geq 0$$
$$y \geq 0$$
$$3x + 2y \leq 6$$
$$-2x + 4y \leq 8$$

Solution: First you sketch the graphical solution S of the given constraints using the methods of Section 10.2.

This bounded region S is also called the **region of feasible solution** or **feasible region**.

Any point in the interior or on the boundary of S satisfies all the above constraints.

Next you find a point (x, y) of the feasible region that gives the maximum value of the objective function Z. Let's first draw some lines which represent the objective function for values of Z = 0, 5, 10 and 15; i.e., the lines

$2x + 5y = 0$ $\quad$ $2x + 5y = 10$

$2x + 5y = 5$ $\quad$ $2x + 5y = 15$

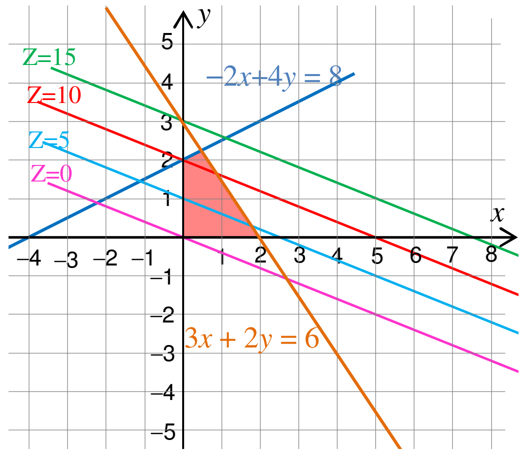


Figure 10.8

From Figure 10.8 you can observe that as the value of Z increases, the lines are moving upwards and the line for Z = 15 is outside the feasible region. The maximum possible value of Z will be obtained if we draw a line between Z = 10 and Z = 15 parallel to them that just "touches" the feasible region.

This occurs at the vertex (corner point) P which is the point of intersection of the lines

$$\left.\begin{aligned} 3x+2y&=6 \\ -2x+4y&=8 \end{aligned}\right\} \Rightarrow x=\frac{1}{2} \text{ and } y=\frac{9}{4}$$

The value of Z at this point is

$$Z = 2x + 5y = 2\left(\frac{1}{2}\right) + 5\left(\frac{9}{4}\right) = \frac{49}{4} = 12\frac{1}{4}$$

Thus the maximum value of Z under the given conditions is $Z = 12\frac{1}{4}$.

As a generalization of this example, we state the following:

Fundamental theorem of linear programming

Theorem 10.1

If the feasible region of a linear programming problem is **non-empty** and **bounded**, then the objective function attains both a maximum and a minimum value and those occur at corner points of the feasible region. If the feasible region is **unbounded**, then the objective function may or may not attain a maximum or minimum value; however, if it attains a maximum or minimum value, it does so at corner points.

Steps to solve a linear programming problem by the graphical method

1. Draw the graph of the feasible region.
2. Compute the coordinates of the corner points.
3. Substitute the coordinates of the corner points into the objective function to see which gives the optimal value.
4. If the feasible region is unbounded, this method is misleading: optimal solutions always exist when the feasible region is bounded, but may or may not exist when it is unbounded.

To apply this to Example 1, we find the vertex points (0, 0), (2, 0), $\left(\frac{1}{2}, \frac{9}{4}\right)$ and (0, 2) and test their values as shown in the following table.

Vertex Point	Value of Z = 2x + 5y
(0, 0)	Z = 2(0) + 5 (0) = 0
(2, 0)	Z = 2 (2)+ 5 (0) = 4
$\left(\frac{1}{2}, \frac{9}{4}\right)$	$Z = 2\left(\frac{1}{2}\right) + 5\left(\frac{9}{4}\right) = \frac{49}{4}$
(0, 2)	Z = 2 (0) + 5 (2) = 10

Comparing the values of Z, you get the maximum value of $Z = \frac{49}{4}$ obtained at $\left(\frac{1}{2}, \frac{9}{4}\right)$.

We also have the minimum value Z = 0 at (0, 0).

Example 2 Solve the following linear programming problem. Find the maximum value of the objective function Z = 3*x* + 2*y*, subject to the following constraints:

$$x \geq 0$$
$$y \geq 0$$
$$x + 2y \leq 4$$
$$x - y \leq 1$$

Solution: From the constraints you sketch the feasible region shown in Figure 10.9.

The vertices of this region are (0, 0), (1, 0), (2, 1) and (0, 2).

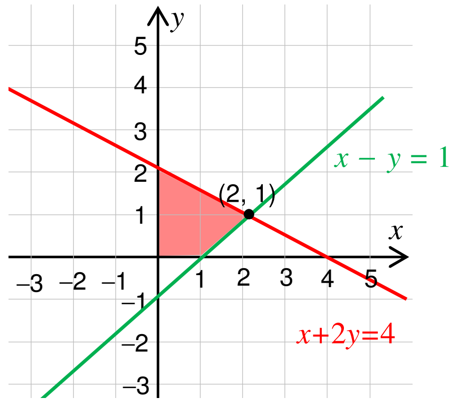


Figure 10.9

Their functional values at Z are given in the following table:

Vertex	Value of Z = 3x + 2y
(0, 0)	$Z = 3(0) + 2(0) = 0$
(1, 0)	$Z = 3(1) + 2(0) = 3$
(2, 1)	$Z = 3(2) + 2(1) = 8$
(0, 2)	$Z = 3(0) + 2(2) = 4$

Thus, the maximum value of Z is 8, and occurs when $x = 2$ and $y = 1$.

ACTIVITY 10.3

1 In Example 2, take some points inside the region S and show that their corresponding values of Z are less than 8.

2 Find the maximum and minimum values of

a Objective function:

$Z = 6x + 10y$

Subject to: $x \geq 0$

$y \geq 0$

$2x + 5y \leq 10$

b Objective function:

$Z = 4x + y$

Subject to: $x \geq 0$

$y \geq 0$

$x + 2y \leq 40$

$2x + 3y \leq 72$

Example 3 Solve the following linear programming problem.

Find the maximum value of $Z = 4x + 6y$ subject to the following constraints:

$$x \geq 0$$
$$y \geq 0$$
$$-x + y \leq 11$$
$$x + y \leq 27$$
$$2x + 5y \leq 90$$

Solution The feasible region bounded by the constraints is shown in Figure 10.10. The vertices of the feasible region are (0, 0), (27, 0), (15, 12), (5, 16) and (0, 11).

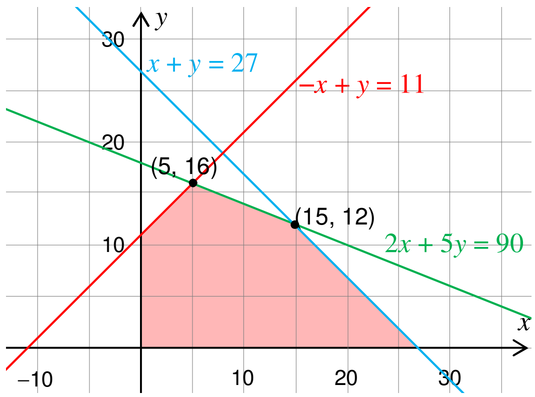


Figure 10.10

Testing the objective function at the vertices gives

Vertex	Value of $Z = 2x + 4y$
(0, 0)	$Z = 4\,(0) + 6\,(0) = 0$
(27, 0)	$Z = 4\,(27) + 6(0) = 108$
(15, 12)	$Z = 4\,(15) + 6\,(12) = 132$
(5, 16)	$Z = 4\,(5) + 6\,(16) = 116$
(0, 11)	$Z = 4\,(0) + 6\,(11) = 66$

Thus the maximum value of Z is 132 when $x = 15$ and $y = 12$.

Example 4 Find values of x and y which minimize the value of the objective function

$Z = 2x + 4y$, subject to: $x \geq 0$

$$y \geq 0$$
$$x + 2y \geq 10$$
$$3x + y \geq 10$$

Solution: From the given constraints the feasible region S is as shown in Figure 10.11.

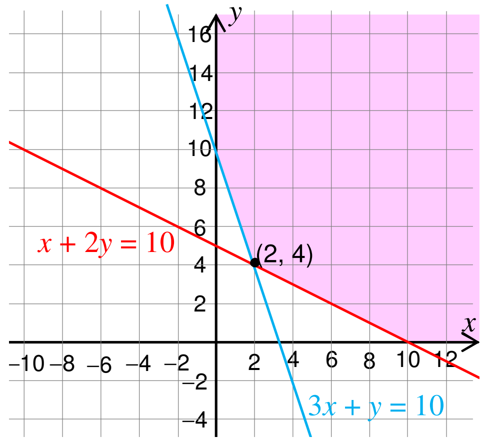


Figure 10.11

This region S is unbounded. The vertices are at (0, 10), (2, 4) and (10, 0) with values given below.

Vertex	Value of Z
(0, 10)	2 (0) + 4 (10) = 40
(2, 4)	2 (2) + 4 (4) = 20
(10, 0)	2 (10) + 4(0) = 20

Here vertices (2, 4) and (10, 0) give the minimum value $Z = 20$ so that the solution is not unique. In fact every point on the line segment through (2, 4) and (10, 0) gives the same minimum value of $Z = 20$.

From this example we can observe that

i an optimization problem can have infinite solutions.

ii not all optimization problems have a solution, since the above problem does not have a maximum value for Z.

Example 5 Find values of x and y that maximize

$$Z = x + 3y, \text{ subject to: } 2x + 3y \leq 24$$
$$x - y \leq 7$$
$$y \leq 6$$
$$x \geq 0$$
$$y \geq 0$$

Solution In Figure 10.12 we have drawn the feasible region of this problem. Since it is bounded, the maximum value of Z is attained at one of five extreme points. The values of the objective function at the five extreme points are given in the following table.

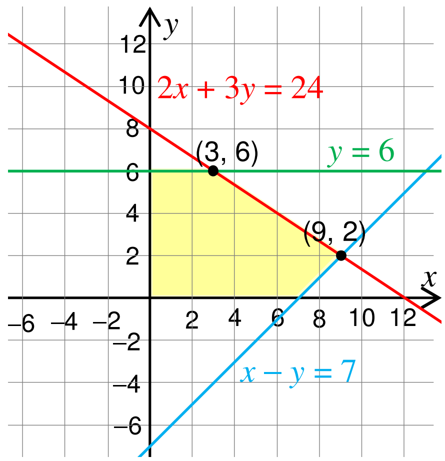


Figure 10.12

Corner point (x, y)	Value of $Z = x + 3y$
(0,6)	18
(3,6)	21
(9,2)	15
(7,0)	7
(0,0)	0

From this table the maximum value of Z is 21, which is attained at $x = 3$ and $y = 6$.

Example 6 Find values of x and y that minimize

$$Z = 2x - y, \text{ subject to: } 2x + 3y = 12$$

$$2x - 3y \geq 0$$

$$x, y \geq 0$$

Solution: In Figure 10.13, we have drawn the feasible region of this problem. Because one of the constraints is an equality constraint, the feasible region is a straight line segment with two extreme points. The values of Z at the two extreme points are given in the following table.

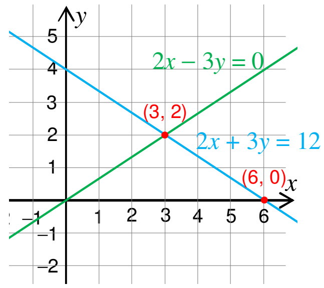


Figure 10.13

Extreme point (x, y)	Value of $Z=2x-y$
(3,2)	4
(6,0)	12

Thus the minimum value of $Z = 2x - y$ is 4 a attained at (3, 2)

Example 7 Maximize $Z = 2x + 5y$ subject to: $2x + y \geq 8$

$$-4x + y \leq 2$$

$$2x - 3y \leq 0$$

$$x, y \geq 0$$

Solution: The feasible region is illustrated in Figure 10.14. Since it is unbounded, we are not assured by Theorem 10.1 that the objective function attains a maximum value. In fact, it is easily seen that since the feasible region contains points for which both x and y are arbitrarily large and positive, the objective function can be made arbitrarily large and positive. This problem has no optimal solution. Instead, we say the problem has an unbounded solution.

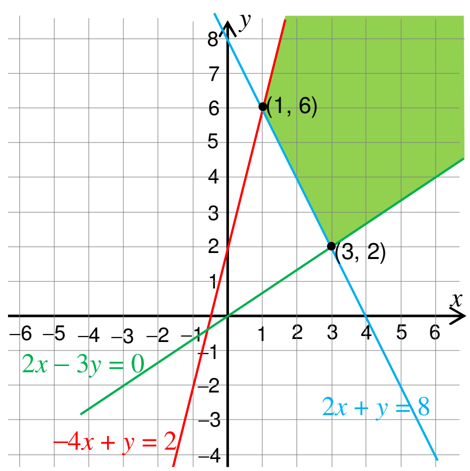


Figure 10.14

Exercise 10.3

Find the maximum and minimum values of

a $Z = 2x + 3y$, subject to: $x \geq 0$, $y \geq 0$, $2y + x \leq 16$, $x - y \leq 10$

b $Z = 2x + 3y$, subject to: $x \geq 0$, $y \geq 0$, $3x + 7y \leq 42$, $x + 5y \leq 22$

c $Z = 4x + 2y$, subject to: $x \geq 0$, $y \geq 0$, $x + 2y \geq 4$, $3x + y \geq 7$, $-x + 2y \leq 7$

d $Z = 4x + 5y$ subject to: $x \geq 0$, $y \geq 0$, $2x + 2y \leq 10$, $x + 2y \leq 6$

e $Z = 4x + 3y$ subject to: $x \geq 0$, $y \geq 0$, $2x + 3y \geq 6$, $3x - 2y \leq 9$, $x + 5y \leq 20$

f $Z = 3x + 4y$ subject to: $x \geq 0$, $y \geq 0$, $x + 2y \leq 14$, $3x - y \geq 0$, $x - y \leq 2$

10.4 REAL LIFE LINEAR PROGRAMMING PROBLEMS

Group Work 10.2

1 Consider a furniture shop that sells chairs and tables where the profit per chair is Birr 9 and the profit per table is Birr 7.

a What is the profit from a sale of 6 chairs and 4 tables?

b If the shop has sold x number of chairs and y numbers of tables, what is the profit in terms of x and y?

2 The number of fields a farmer plants with wheat is W and the number of fields with corn is C. The restrictions on the number of fields are that:

a there must be at least 2 fields of corn.

b there must be at least 2 fields of wheat.

c not more than 10 fields in total are to be sown with wheat or corn.

Construct three inequalities from the given information and sketch the region which satisfies the 3 inequalities.

In everyday life, we are often confronted with a need to allocate limited resources to best advantage. We may want to maximize an objective function (such as profit) or minimize (say, cost) under some restrictions (which we called constraints).

Despite the apparently quite restrictive nature of the Linear Programming problem format there are many practical problems in industry, government and other organization which fall into this type. Below we give real life examples of simple linear Programming problems, each of which represents a classic type of Linear Programming problem.

Example 1 A manufacturer wants to maximize the profit for two products. Product I gives a profit of Birr 1.50 per kg, and product II gives a profit of Birr 2.00 per kg. Market tests and available resources have indicated the following constraints.

a The combined production level should not exceed 1200 kg per month.

b The demand for product II is not more than half the demand for product I.

c The production level of product I is less than or equal to 600 kg plus three times the production level of product II.

Find the number of kg of each product that should be produced in a month to maximize profit.

Solution: The first step in solving such real life Linear Programming problems is to assign variables to the numbers to be determined for a maximum (or a minimum) value of the objective function.

Let x = the number of kg of product I, and

y = the number of kg of product II

These variables are usually called **decision variables**.

The objective of the manufacturer is to decide how many units of each must be produced to maximize the objective function (profit) given by:

$$P = 1.5x + 2y$$

The above three constraints can be translated into the following linear inequalities

a $x + y \le 1200$

b $y \le \frac{1}{2}x$ or $-x + 2y \le 0$

c $x \le 3y + 600$ or $x - 3y \le 600$

Since neither x nor y can be negative, we have the additional non-negativity constraints of $x \ge 0$ and $y \ge 0$. The above information can now be transformed into the following linear programming problem.

Maximize $P = 1.5x + 2y$

Subject to $x \geq 0$

$y \geq 0$

$x + y \leq 1200$

$-x + 2y \leq 0$

$x - 3y \leq 600$

The constraints above have region of feasible solutions shown in Figure 10.15.

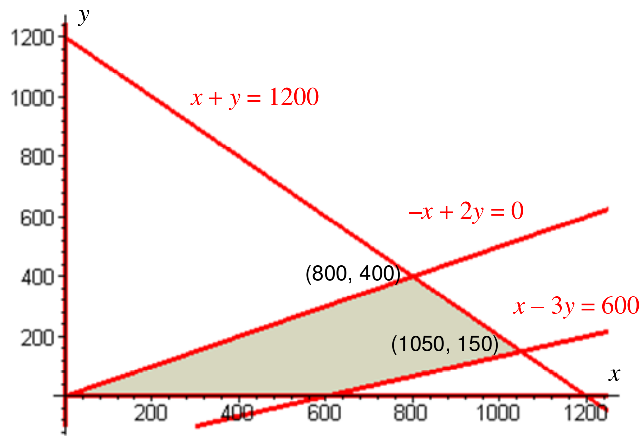


Figure 10.15

To solve the maximization problem geometrically, we first find the vertices by finding the points of intersection of the border lines of S, to get

O (0, 0), A (600, 0), B (1050, 150) and C (800, 400)

Then a solution can be obtained from the table below:

Vertex	Profit $P = 1.5x + 2y$
O (0, 0)	P = 1.5 (0) + 2(0) = 0
A (600, 0)	P = 1.5 (600) + 2(0) = 900
B (1050, 150)	P = 1.5 (1050) + 2 (150) = 1875
C (800, 400)	P = 1.5 (800) + 2 (400) = 2000

Thus the maximum profit is Birr 2000 and it occurs when the monthly production consists of 800 units of product I and 400 units of product II.

(Observe that the minimum profit is Birr 0 which occurs at the vertex O (0, 0)).

Example 2 A manufacturer of tents makes a standard model and an expedition model for national distribution. Each standard tent requires 1 labour-hour from the cutting department and 3 labour-hours from the assembly department. Each expedition tent requires 2 labour-hours from cutting and 4 labour-hours from assembly. The maximum labour-hours available per day in the cutting department and the assembly department are 32 and 84, respectively. If the company makes a profit of Birr 50.00 on each standard tent and Birr 80 on each expedition tent, how many tents of each type should be manufactured each day to maximize the total daily profit? (Assume that all tents produced can be sold.)

Solution: The information given in the problem can be summarized in the following table.

	Labour-hr per tent		Max. Labour-hr per day
	Standard	Expedition	
Cutting dept	1	2	32
Assembly dept	3	4	84
Profit	Birr 50	Birr 80	

Then we assign decision variables as follows:

Let x = number of standard tents produced per day

y = number of expedition tents produced per day

The objective of management is to decide how many of each tent should be produced each day in order to maximize profit P = $50x + 80y$

Both cutting and assembly departments have time constraints given by

$1\times x + 2\times y \leq 32$ . . . . .cutting dept. constraint

$3\times x + 4\times y \leq 84$ . . . . assembly dept. constraint

where $x \geq 0$ and $y \geq 0$ . . . . non-negative constraints

The Linear Programming problem is then to maximize P = $50x + 80y$,

subject to: $x + 2y \leq 32$

$3x + 4y \leq 84$

$x, y \geq 0$

To get a graphical solution, we have the feasible region S shown in Figure 10.16. The vertices are at (0, 0), (28, 0), (20, 6) and (0, 16). The maximum value of profit can be obtained from the following table.

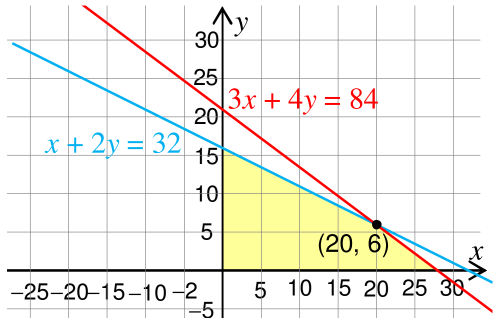


Figure 10.16

Vertex	Value of P = 50*x* + 80*y*
(0, 0)	$P = 50(0) + 80(0) = 0$
(28, 0)	$P = 50\ (28) + 80\ (0) = 1{,}400$
(20, 6)	$P = 50\ (20) + 80\ (6) = 1{,}480$
(0, 16)	$P = 50\ (0) + 80\ (16) = 1{,}280$

Thus the maximum profit of Birr 1,480 is attained at (20, 6); i.e. the manufacturer should produce 20 standard and 6 expedition tents each day to maximize profit.

Example 3 A patient in a hospital is required to have at least 84 units of drug A and 120 units of drug B each day. Each gram of substance M contains 10 units of drug A and 8 units of drug B, and each gram of substance N contains 2 units of drug A and 4 units of drug B. Suppose both substances M and N contain an undesirable drug C, 3 units per gram in M and 1 unit per gram in N. How many grams of each substance M and N should be mixed to meet the minimum daily requirements and at the same time minimize the intake of drug C? How many units of drug C will be in this mixture?

Solution Let us summarize the above information as:

	Substance M	Substance N	Min-requirement
Drug A	10	2	84
Drug B	8	4	120
Drug C	3	1	

Let x = number of grams of substance M used

y = number of grams of substance N used

Our objective is to minimize drug C from $3x + y$.

The constraints are the minimum requirements of

$10x + 2y \geq 84$ . . . . from drug A

and $8x + 4y \geq 120$ . . . . from drug B

Since both M and N must be non negative $x \geq 0$, $y \geq 0$.

Thus our optimization problem is to

Minimize $C = 3x + y$,

subject to : $10x + 2y \geq 84$

$8x + 4y \geq 120$

$x, y \geq 0$

The sketch of the feasible region S is given in Figure 10.17.

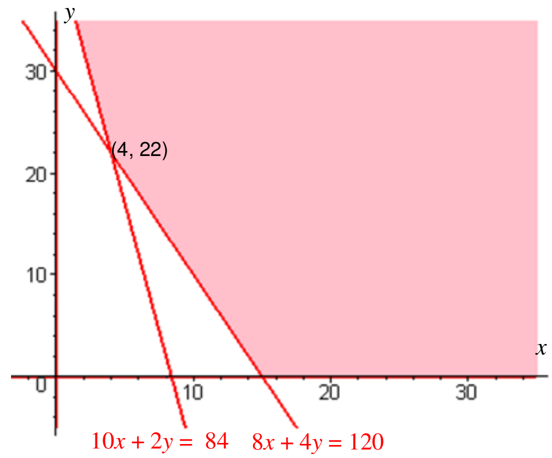


Figure 10.17

To obtain the minimum value graphically, we use the table

Vertex	Value of $C = 3x + y$
(0, 42)	C = 3(0) + 42 = 42
(4, 22)	C = 3 (4) + 22 = 34
(15, 0)	C = 3(15) + 0 = 45

The minimum intake of drug C is 34 units and it is attained at an intake of 4 grams of substance M and 22 grams of substance N.

We can summarize the steps in solving real life optimization problems geometrically as follows.

Step 1: Summarize the relevant information in the problem in table form.

Step 2: Form a mathematical model of the problem by introducing decision variables and expressing the objective function and the constraints using these variables.

step 3: Graph the feasible region and find the corner points.

Step 4: Construct a table of the values of the objective function at each vertex.

Step 5: Determine the optimal value(s) from the table.

Step 6: Interpret the optimal solution(s) in terms of the original real life problem.

Exercise 10.4

Solve each of the following real life problems:

a A farmer has Birr 1,700 to buy sheep and goats. Suppose the unit price of sheep is Birr 300 and the unit price of goats is Birr 200.

- **i** If he decided to buy only goats, what is the maximum number of goats he can buy?
- **ii** If he has bought 2 sheep what is the maximum number of goats he can buy with the remaining money?
- **iii** Can the farmer buy 4 sheep and 3 goats? 2 sheep and 5 goats? 3 sheep and 4 goats?

b A company produces two types of tables; Tables A and Table B. It takes 2 hours of cutting time and 4 hours of assembling to produce Table A. It takes 10 hours of cutting time and 3 hours of assembling to produce Table B. The company has at most 112 hours of cutting labour and 54 hours of assembly labour per week. The company's profit is Birr 60 for each Table A produced and Birr 170 for each Table B produced. How many of each type of table should the company produce in order to maximize profit?

c The officers of a high school senior class are planning to rent buses and vans for a class trip. Each bus can transport 36 students, requires 4 supervisors and costs Birr 1000 to rent. Each van can transport 6 students, requires 1 supervisor, and costs Birr150 to rent. The officers must plan to accommodate at least 420 students. Since only 48 parents have volunteered to serve as supervisors, the officers must plan to use at most 48 supervisors. How many vehicles of each type should the officers rent in order to minimize the transportation costs? What is the minimum transportation cost?

Key Terms

bounded solution region	**minimum value**
constraints	**objective function**
decision variables	**optimal value**
equation of a line	**real life linear programming problems**
Fundamental theorem of linear programming	**slope of a line**
half planes	**solution region**
inclination of a line	**system of linear inequalities**
maximum value	**vertex (corner point)**

Summary

1 The **angle of inclination** of a line L is the angle α measured from the x-axis to L in the counter clock wise direction.

2 The **slope** of a line passing through P(x, y) and Q(x_2, y_2) is

$$m = \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1}, \text{ for } x_1 \neq x_2.$$

3 If a line has slope m and passes through P(x_1, y_1), the slope-point form of its **equation** is given by $y - y_1 = m(x - x_1)$

4 An equation of a line can be reduced to the form $ax + by = c$. $a, b, c \in \mathbb{R}$ with $a \neq 0$ or $b \neq 0$.

5 A line ℓ divides the plane into two **half-planes.**

6 A **system of linear inequalities** is a collection of two or more linear inequalities to be solved simultaneously.

7 A **graphical solution** is the collection of all points that satisfy the system of linear inequalities.

8 A **vertex** (or **corner point**) of a solution region is a point of intersection of two or more boundary lines.

9 A solution region is said to **bounded**, if it can be enclosed in a rectangle.

10 A number $M = f(c)$ for c in I is called the **maximum value** of f on I, if $M \geq f(x)$ for all x in I.

11 A number $m = f(d)$ for d in I is called the **minimum value** of f on I, if $m \leq f(x)$ for all x in I.

12 A value which is either a maximum or a minimum value is called an **optimal** (or **extremum**) value.

13 An optimization problem involves maximizing or minimizing an **objective function** subject to **constraints**.

14 If an optimal value of an objective function exists, it will occur at one or more of the corner points of the feasible region.

15 In solving real life linear programming problems, assign variables called **decision variables.**

Review Exercises on Unit 10

1 Find the slope of the line

a that passes through the points P(4,−1) and Q(0, 2)

b that has angle of inclination $\alpha = 45^{\circ}$

c that is parallel to the line $2x - 4y = 6$

2 Draw the graphs of the lines ℓ_1: $y = 3x - 4$ and ℓ_2: $x - 5y = 2$ using the same coordinate axes.

3 Find graphical solutions for each of the following systems of linear inequalities.

a $x - 5y \leq 2$
$3x - y \leq 4$

b $y + 2x \geq 4$
$y - 2x > 4$

c $x \geq 2$
$y \geq 0$
$x + y \leq 5$

d $x \geq 0$
$y \geq 0$
$3x + 2y < 6$

4 Find the maximum and minimum values of the objective function subject to the given constraints.

a Objective function $Z = 3x + 2y$,
subject to: $x \geq 0$
$y \geq 0$
$x + 3y \leq 15$
$4x + y \leq 16$

b Object function $Z = 2x + 3y$,
subject to: $x \geq 0$
$y \geq 0$
$2x + y \geq 100$
$x + 2y \geq 80$

c Objective function $Z = 10x + 7y$
subject to: $0 \leq x \leq 60$
$0 \leq y \leq 45$
$5x + 6y \leq 420$

d Objective function $Z = 3x + 4y$,
subject to: $x \geq 1$,
$y \geq 0$
$3x - 4y \leq 12$
$x + 2y \geq 4$

5 Find the optimal solution of the following real life linear programming problems.

a Ahadu company produces two models of radios. Model A requires 20 min of work on assembly line I and 10 min of work on assembly line II. Model B requires 10 min of work on assembly line I and 15 min of work on assembly line II. At most 22 hrs of assembly time on line I and 25 hrs of assembly time on line II are available per week. It is anticipated that Ahadu company will realize a profit of Birr 10 on model A and Birr 14 on model B. How many radios of each model should be produced per week in order to maximize Ahadu's profit?

b A farming cooperative mixes two brands of cattle feed. Brand X costs Birr 25 per bag and contains 2 units of nutritional element A, 2 units of nutritional element B, and 2 units of element C. Brand Y costs Birr 20 per bag and contains 1 unit of nutritional element A, 9 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 12, 36 and 24 units, respectively. Find the number of bags of each brand that should be mixed to produce a mixture having a minimum cost.

Unit 11

MATHEMATICAL APPLICATIONS IN BUSINESS

Unit Outcomes:

After completing this unit, you should be able to:

- *know common terms related to business.*
- *know basic concepts in business.*
- *apply mathematical principles and theories to practical situations.*

Main Contents:

11.1 BASIC MATHEMATICAL CONCEPTS IN BUSINESS

11.2 COMPOUND INTEREST AND DEPRECIATION

11.3 SAVING, INVESTING, AND BORROWING MONEY

11.4 TAXATION

Key terms

Summary

Review Exercises

INTRODUCTION

In this unit you will learn the basic mathematical concepts in business and the techniques of computing compound interest. Furthermore, you will observe how and why money is saved, invested and borrowed. At the end, the concept of tax, the reason why people should pay tax and how to calculate it are discussed.

This unit has four sections. The first section deals with the concept of ratio, rate, proportion, and percentage. Here you will see how these concepts are implemented in business. The second section deals with the computation of compound interest, ordinary annuity, and depreciation of a fixed asset. The third section deals with the concepts of saving, investing, and borrowing money. The fourth section deals with taxation and the different types of taxes commonly implemented in Ethiopia. Each section deals with solving problems that are associated with business activities.

OPENING PROBLEM

Yilma obtained a gift of 10,000 Birr from his grandmother on his first birthday. His parents decided to deposit his money in the Commercial Bank of Ethiopia for his University education. It is noted that the bank pays an interest rate of 4% compounded semiannually. If Yilma's parents deposit the money on his first birth date, how much will he obtain when he joins the university at the age of 18 years exactly? What is the amount of interest his money has earned?

11.1 BASIC MATHEMATICAL CONCEPTS IN BUSINESS

The concepts of ratio, rate, proportion and percentage are widely used whenever we deal with business in our daily live activities. Hence, we will look at each of these concepts and their applications in this section.

A Ratio

Consider the following two questions:

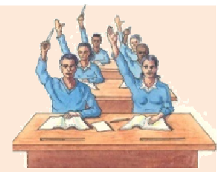
Question 1 *How many students are there in your school?*

Question 2 *How many teachers are there in your school?*

Compare your answer with the explanation given below.

Suppose the number of students and teachers in a given school are 3900 and 75, respectively. From this we can make the statement that "the ratio of teachers to students in the school is 1 to 52" or we can say that "the ratio of students to teachers in the school is 52 to 1". This tells us that for every 52 students in the school there corresponds one teacher.

ACTIVITY 11.1



Out of 60 students in a class 20 are boys. What is

a the ratio of boys to girls?

b the ratio of boys to the students in a class?

A ratio of a to b is expressed as a: b, or $a \div b$ or $\frac{a}{b}$ for $b \neq 0$.

The numbers appearing in a ratio are called **terms** of the ratio and they must be expressed in the same unit of measurement.

A ratio can be expressed in one of two ways:

i part-to-whole ratio or **ii** part-to-part ratio

ii
iii

Definition 11.1

A **ratio** is a comparison of two or more quantities expressed in the same unit of measurement.

Example 1 The following table gives the number of teachers in a given school according to their education level and sex.

	Diploma holders	Degree Holders	Total
Male	26	46	72
Female	16	12	28
Total	42	58	100

a What is the ratio of female diploma holders to the number of teachers in the school?

b What is the ratio of diploma holders to degree holders in the school?

Solution:

a The first question is asking the part-to-whole ratio, hence it is 16:100 or 4:25.

b The second question is asking the part-to-part ratio, hence it is 42:58 or 21:29.

✍Note:

The value of a ratio is usually expressed in its lowest terms.

Example 2 What is the ratio of 1.6 meters to 180 centimetres?

Solution: To compare two measurements in different units you must change one of the units of measurement to the other unit.

If you change 1.6 meters to centimetres; we get

1 meter = 100 centimetres $\Rightarrow$ 1.6 meter = 160 cms

Therefore, the ratio is $\frac{160\text{ cms}}{180\text{ cms}} = \frac{8}{9}$ or 8: 9

Similarly, if we change 180 centimetres to the unit of meters:

$180\text{ cm} = \frac{180\text{ cm}\times 1\text{ m}}{100\text{ cm}} = 1.8\text{ m}$. Therefore, the ratio is $\frac{1.6\text{ m}}{1.8\text{ m}} = \frac{16}{18} = \frac{8}{9}$ or 8: 9.

Note that, in both cases, the ratio is the same (8: 9).

People commonly form a group and involve on a given business activity according to their individual contribution for the business. In this case, their individual profit is allocated according to the ratio of their investment.

Example 3 Allocate Birr 1500 in the ratio 2:3:7.

Solution Note that the terms in the ratio are positive integers. First, you need to determine the total number of parts to be allocated.

That is 2 + 3 + 7 = 12.

Now determine the value of each single part, which is obtained by dividing the total amount by the total parts to be allocated: $\frac{1500}{12}$ = Birr 125 per part.

To allocate, multiply each term of the ratio by the value of the single part, i.e. $2\times125=250$, $3\times125=375$, and $7\times125=875$.

Therefore, the allocation will be Birr 250, Birr 375, and Birr 875, respectively.

Example 4 Allocate Birr 800 among three workers in the ratio of $\frac{2}{3}:\frac{1}{4}:\frac{1}{2}$.

Solution If the terms of the ratio are fractions, they must be converted to equivalent fractions with the same denominator and the amount is then allocated in the ratio of the numerators. So that

$$\frac{2}{3}:\frac{1}{4}:\frac{1}{2}=\frac{8}{12}:\frac{3}{12}:\frac{6}{12}$$

Determine the total number of parts by adding the numerators: 8 + 3 + 6 = 17.

Then the value of a single part is Birr $\frac{800}{17}$.

Then allocate according to the ratio of the numerators to each:

$8\times\frac{800}{17}$ = Birr 376.47, $3 \times \frac{800}{17}$ = Birr 141.18 and $6 \times \frac{800}{17}$ = Birr 282.35.

Therefore, the allocation will be Birr, 376.47 Birr, 141.18 and Birr, 282.35 respectively.

Exercise 11.1

1. A profit of Birr 19,560 is to be divided between four partners in the ratio of 3:2:1:6. How much should each receive?
2. A sum of money was divided between Aster, Ali, and Mesfin in the ratio of $\frac{2}{5}:\frac{4}{3}:2$, respectively. Aster has received Birr 3504. How much money was there to start with?

B Rates

In construction activity one has to know the ratio at which the amount of cement, sand and gravel are mixed to form the appropriate mixture required for specified purpose. For example, to make a beam or a column of residential building, cement, sand and gravel are mixed in the ratio 1:2:3, respectively. In this case cement is measured in quintals while sand and gravel are measured using a $0.4\times0.4\times0.2$ cubic meter box. Hence the ratio involves different units of measurement and this will lead us to the following definitions.

Definition 11.2

A **rate** is a comparison of two or more quantities expressed in different units of measurement.

There are a number of situations where one wishes to compare "unlike quantities" such as the ratio of kilometres travelled per litter of gasoline, the amount of product produced per hour in a given factory, and so on.

✍Note:

A ratio can be a rate.

Example 5 The distance from Addis Ababa to Adama is 100 km. Ahmed travelled by minibus from Addis Ababa to Adama early in the morning and it took him 1 hour and 20 minutes. What is the rate of speed of his journey?

Solution The rate of speed of his journey is the ratio of the distance travelled and the time it took. Since the distance is 100 km and the time taken is $\frac{4}{3}$ hours, the rate is:

$$100 \text{ kms} : \frac{4}{3} \text{ hrs} = \frac{100 \text{ kms}}{\frac{4}{3} \text{ hrs}} = \frac{300}{4} \text{ kms per hr} = 75 \text{ kms/hr}$$

Example 6 Five tyre-repairers are working in a group and fixed 210 tyres in a given day of the week. What is the rate of tyres fixed per person?

Solution Total number of tyres fixed is 210 and the number of workers involved is 5. Hence, the rate per person will be the ratio of the number of tyres fixed to the number of workers involved, i.e.,

$$210: 5 = \frac{210}{5} = 42 \text{ tyres per person.}$$

In dealing with business, production, population, and so on, it is common to describe by what amount a quantity has increased or decreased based on some starting or fixed level. This will lead us to the rate of change of a given quantity given by the relation:

$$\text{Rate of change } = \frac{\text{amount of change}}{\text{original amount}} = \frac{\text{final amount -original amount}}{\text{original amount}}$$

The rate of change will be a rate of increase if the amount of change is positive and a rate of decrease if the amount of change is negative.

Example 7 The price of a quintal of cement in Addis Ababa in September 2008 was Birr 220, and ten months later, on July 2009, its price was Birr 370. What is the rate of increase in the price of one quintal of cement from September 2008 to July 2009?

Solution We are given that: the original price = Birr 220 and the new price = Birr 370. Hence change in price = Birr 370 – Birr 220 = Birr 150

$$\text{Rate of increase} = \frac{\text{amount of increase}}{\text{original amount}} = \frac{150}{220} = 0.682$$

Example 8 Aster has invested 20,000 Birr in a fruit wholesaler. A year later the audit report on the business indicated that there was 16,200 Birr as a balance. Find the rate of decrease that resulted in one year.

Solution Since the balance indicated that there is a decrease from the amount of capital invested, we have a decrease rate.

$$\text{Rate of decrease } = \frac{\text{amount of decrease}}{\text{original investment}} = \frac{16,200 - 20,000}{20,000} = -0.19$$

The negative sign indicates that there is a decrease in the investment which is a loss.

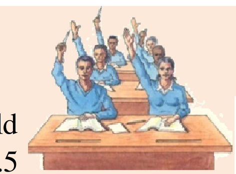
Exercise 11.2

1 A carpenter's daily production of school chairs increased from 20 units to 40 units. At the same time his income (or revenue) increased from 1600 Birr to 2400 Birr. What is the rate of change of income per unit?

2 A steel company has imported 35 tons of raw material from South Africa in 1995. In 2008 the company imported 54 tons of raw material from the same country. What is the rate of change of amount imported?

C Proportion

ACTIVITY 11.2



A combine harvester machine can harvest three hectares of wheat field in one hour at a rate of 150 Birr per hour. If a farmer owns 16.5 hectares of wheat field how much does he pay to harvest his wheat?

Definition 11.3

A **proportion** is a statement of equality between two ratios.

For $a, b, c, d \in \mathbb{R}$, with $b \neq 0$ and $d \neq 0$, one way of denoting a proportion is $a : b = c : d$, which is read as "a is to b" as "c is to d". Of course, by definition, $\frac{a}{b} = \frac{c}{d}$, which means that a proportion is an equation between two ratios.

In the proportion $a : b = c : d$, with $b \neq 0$ and $d \neq 0$, the four numbers are referred as the **terms** of the proportions. The first and the last terms a and d are called the **extremes**; the second and third terms b and c are called the **means**. In the proportion $a : b = c : d$, the product of the extremes is equal to the product of the means; that is,

$$\frac{a}{b} = \frac{c}{d} \text{ is equivalently represented as } a \cdot d = b \cdot c.$$

For three quantities a, b and c such that $\frac{a}{b} = \frac{b}{c}$, which is equivalent to $b^2 = a \cdot c$, b is called the **mean proportional** between a and c.

Example 9 On a residence plan of Ato Admasu, 1 cm on the plan represents 150 cms on the ground. Find the distance on the ground for the distance represented by 3.20 cms on the plan.

Solution On the map we have the ratio 1:150. Let x be the distance on the ground. Then the distance represented by 3.20 cms on the plan can be found by the proportion $\frac{3.20}{x} = \frac{1}{150}$.

Hence, $x = \frac{150\,\text{cm} \times 3.20\,\text{cm}}{1\,\text{cm}} = 480\,\text{cms}$ on the ground.

Example 10 A secretarial pool (15 secretaries in all) on one floor of a corporate complex has access to 11 telephones. If on a different floor, there are 23 secretaries, approximately what number of telephones should be available?

Solution Let x be the number of telephones available on the other floor. Then we have the proportion 15:11 = 23: x, that is, $\frac{15}{11} = \frac{23}{x}$.

Hence $x = \frac{11 \times 23}{15} = 16.87$. Therefore, 17 telephones are required.

Compound proportion

From the above discussion you have seen how change in one variable quantity depends on a change in another variable quantity (i.e., simple proportion). However, the value of a variable quantity most often depends on the value of two or more other variable quantities. For example,

- ✓ The cost of sheet metal depends on the area of the sheet, thickness of the sheet, and the cost per unit area of the metal.
- ✓ The amount of interest obtained depends on the amount of money deposited in a bank, length of time it is deposited, and rate of interest per year.
- ✓ The amount of product produced depends on the amount of capital and labour hour units used.

Definition 11.4

A **compound proportion** is a situation in which one variable quantity depends on two or more other variable quantities. Specifically, if a variable quantity y is proportional to the product of two or more variable quantities, we say that y is **jointly proportional** to these variable quantities, or y varies jointly as these variables.

If z is jointly proportional to x and y (or z is proportional to x and y), then in short we write it as $z \propto xy$. Its equivalent representation in terms of an equation is $z = kxy$, where k is a constant of proportionality.

Note that in a compound proportion, a proportion combination of direct and/or inverse variation may occur. If z is directly proportional to x and inversely proportional to y, then we can write it as $z \propto \frac{x}{y}$ or equivalently $z = k\frac{x}{y}$, where k is a constant of proportionality.

Example 11 If z is proportional to y and to the square of x and $z = 80$ when $x = 2$ and $y = 5$, then find the equation that relates the variables $x, y,$ and z.

Solution We are given that $z \propto yx^2$ which is equivalent to $z = kyx^2$, where k is a constant of proportionality

To determine the constant of proportionality, put the given values of the variables.

$$80 = k(5)(2^2) = 20k.$$

Hence, $k = 4$. Therefore the equation that relates the three variables is $z = 4yx^2$.

Example 12 The power (P) of an electric current varies jointly as the resistance (R) and the square of the current (I). Given that the power is 12 watts when the current is 0.5 amperes and the resistance is 40 ohms, find the power if the current is 2 amperes and the resistance is 20 ohms.

Solution $P \propto RI^2$, that is, $P = kRI^2$, where k is a constant of proportionality.

Putting the given values in the equation and solving for k, we have

$$12 = k(40)(0.5)^2 \Rightarrow k = \frac{12}{(40)(0.5)^2} = 1.2.$$

Hence the relationship between the three variables is $P = 1.2RI^2$ and the required power is

$$P = 1.2(20)(2)^2 = 96 \text{ watts}.$$

D Percentage

ACTIVITY 11.3

In a class of 60 students 5 of them were absent in a given day.
What percent of the class was absent?

Definition 11.5

A **percentage** is the numerator of a fraction whose denominator is 100. The term percent is denoted by % which means "per one hundred".

Example 13 Express each of the following fractions as percentages.

a $\frac{4}{5}$ **b** $\frac{5}{200}$ **c** $\frac{61}{50}$

Solution First express the given fractions as decimal numbers and multiply by 100%.

a You know that $\frac{4}{5} = 0.8$. Hence $\frac{4}{5} = 0.8 \times 100\% = 80\%$.

b $\frac{5}{200} = 0.025$. Hence $\frac{5}{200} = 0.025 \times 100\% = 2.5\%$.

c If you divide 61 by 50 you will have, $\frac{61}{50} = 1.22$.

Hence $\frac{61}{50} = 1.22 \times 100\% = 122\%$.

When percentages are involved in computations, the corresponding decimal representation is usually used. Percentage is obtained by multiplying a number called the **base** by the **percent**, called the **rate**.

Percentage = base × rate

Consider the following examples to have better understanding of percentage and how it can be applied to solve practical problems.

Example 14

a Find 3% of Birr 57? **b** Find $3\frac{1}{2}\%$ of Birr 900?

Solution

a To find 3% of Birr 57, the base is 57 and the rate is 3% = 0.03, then

Percentage = base × rate = $57 \times \frac{3}{100} = \frac{171}{100} =$ Birr 1.71.

b To find $3\frac{1}{2}\%$ of Birr 900, the base is Birr 900 and the rate is

$3\frac{1}{2}\% = 3.5\% = 0.035$.

Then percentage = base × rate = 9000 × 0.035 = Birr 31.50.

Example 15

a What is the total amount whose 15% is 120?

b Birr 62.50 is what percent of Birr 25,000?

Solution

a Here we are looking for the total amount, whose percentage is 120 and the rate is 0.15. Therefore,

$$\text{Base} = \frac{\text{percentage}}{\text{rate}} = \frac{120}{0.15} = 120 \times \frac{100}{15} = 800 \text{ units.}$$

b Here the base is Birr 25, 000 and the percentage is Birr 62.50.

Hence the rate is

$$\text{Rate} = \frac{\text{percentage}}{\text{base}} = \frac{62.50}{25{,}000} = 0.0025 = \frac{1}{4}\% .$$

Example 16 If the value added tax (VAT) on sales is 15%, find the VAT on a sale of refrigerator that costs Birr 3,800. What is the total cost of the refrigerator?

Solution The rate is 0.15 and the base is Birr 3,800, hence the percentage would be

Percentage = base × rate = 3,800 × 0.15 = Birr 570.

The VAT on the refrigerator is Birr 570.

The total cost of the refrigerator = cost + VAT

= Birr 3,800 + Birr 570

= Birr 4,370.

Commercial Discount

In business activities, it is common to offer a sales discount due to clearance of available stock, changing the business activity, approaching expiry date, and so on. In such cases the discount of an item is described in terms of percentage. For example, you may have 20% discount, 30% discount, and so on.

If p is the original price of an item and r is the percentage of discount, then the amount of discount is given by:

Discount = $r.p$

Therefore, the sales price will be given by:

Discount sales price = Original price − Discount = $p - r.p = p\,(1 - r)$

Example 17 A wool suit, discounted by 30% for a clearance sale, has a price tag of Birr 399. What was the suit's original price? What is the amount of discount?

Solution Let p be the original price of the suit. The amount of discount is $0.30p$. Hence

$$\text{Sales price} = p - 0.30p = 0.70p \Rightarrow 399 = 0.70\,p \Rightarrow p = \frac{399}{0.70} = Birr\ 570$$

Therefore, the original price p = Birr 570 and the amount of discount is

570 – 399 = Birr 171.

Exercise 11.3

1. From 250 candidates who sat for a written examination for a job, 45 of them scored above 85%. The personnel division suggested that those candidates who have scored above 85% in the written examination could sit for interview. What percent of the candidates did not have a chance for interview?
2. A car dealer, at a year-end clearance, reduces the price of last year's models by a certain amount. If a certain four-door model has been sold at a discounted price of Birr 51,000, with a discount of Birr 9,000, what is the percentage of discount?

Markup

In order to make a profit, any institution or company must sell its products for more than the product costs the company to make or buy. The difference between a product selling price and its cost is called **markup**.

Markup = Selling price – Cost price

Example 18 If the price of cement is Birr 250 per quintal and you sell it for Birr 330 per quintal, find the markup per quintal.

Solution Markup = Selling price – Cost

= Birr 330 per quintal – Birr 250 per quintal = Birr 80 per quintal

Markup is usually expressed in terms of percentage with respect to selling price and cost. Markup with respect to selling price is given by;

$$\text{Markup percent} = \frac{\text{markup}}{\text{selling price}} \times 100\%$$

Similarly markup with respect to cost is given by:

$$\text{Markup percent} = \frac{\text{markup}}{\text{cost}} \times 100\%$$

Example 19 If you buy a gold ring for 498 Birr and sell it for 750 Birr, find the markup percent

a with respect to selling price. **b** with respect to cost.

Solution: Markup = Selling price – Cost price = Birr 750 – Birr 498 = Birr 252.

a The markup percent with respect to the selling price is:

$$\text{Markup percent } = \frac{\text{markup}}{\text{Selling price}} \times 100\% = \frac{252}{750} \times 100\% = 33.6\%.$$

b The markup percent with respect to the cost is:

$$\text{Markup percent } = \frac{\text{markup}}{\text{cost price}} \times 100\% = \frac{252}{498} \times 100\% = 50.6\%.$$

Example 20 A merchant wants to sell a semi-automatic washing machine at 3,000.35 Birr with 15% markup on its cost. What is its cost for the merchant?

Solution Given selling price = Birr 3,000.35 and markup percent 15% = 0.15, you need to find cost. But from the relation

$$\text{Markup percent} = \frac{\text{markup}}{\text{cost}} \times 100\%, \text{ we have,}$$

$$\text{Markup percent} = \frac{\text{selling price} - \text{cost}}{\text{cost}} \times 100\%.$$

$$\text{Givng Markup percent } \times \text{cost} = (\text{selling price} - \text{cost}) \qquad (\text{Since } 100\% = 1)$$

$$(\text{Markup percent } + 1) \times \text{cost } = \text{selling price } .$$

$$\text{Hence, cost} = \frac{\text{selling price}}{\text{Markup percent} + 1} = \frac{3,000.35}{0.15 + 1} = \text{Birr } 2,609$$

Example 21 A boutique buys a T-shirt for Birr 54.25 and wants a markup of 30% on retail. What is the selling price?

Solution Given cost = Birr 54.25. Markup percent is 30% on selling price. Then we need to find selling price.

Cost = selling price – markup = 100% – 30% = 70 % of selling price.

This is called the complement of markup percent on selling price.

Hence, the selling price will be:

$$\text{cost} = 0.70 \times \text{selling price } \Rightarrow 54.25 \text{ Birr} = 0.70 \times \text{selling price}$$

$$\Rightarrow \text{selling price} = \frac{54.25}{0.70} = 77.50 \text{ Birr.}$$

In business, it is often necessary to make conversion between percent markups based on cost and selling price. To convert markup percent based on cost to markup percent based on selling price, use the following relation:

$$\text{Markup percent on selling price} = \frac{\text{markup percent on cost}}{\text{selling price (as percent of cost)}} \times 100\%$$

$$= \frac{\text{markup percent on cost}}{100\% + \text{markup percent on cost}} \times 100\%$$

Similarly, to convert markup percent based on selling price to markup percent based on cost, use the relation:

$$\text{Markup percent on cost} = \frac{\text{markup percent on selling price}}{\text{cost (as percent of selling price)}} \times 100\%$$

$$= \frac{\text{markup percent on cost}}{100\% - \text{markup percent on selling price}} \times 100\%$$

Example22 What is the percent markup on selling price if the markup on cost is 25%?

Solution Since we are given the markup on cost, we use the relation

$$\text{Markup percent on selling price} = \frac{\text{markup percent on cost}}{\text{selling price (as percent of cost)}} \times 100\%$$

$$= \frac{\text{markup percent on cost}}{100\% + \text{markup percent on cost}} \times 100\%$$

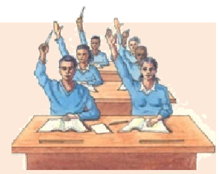
$$= \frac{25\%}{100\% + 25\%} \times 100\% = 20\%$$

Exercise 11.4

1. A pair of shoes costs Birr 110 and sells for Birr 155. Find the markup and the markup percent based on the retail (selling price).
2. What is the percent markup on cost, if the markup on retail is 37%?
3. If W/ro Chaltu purchased a gallon of oil at Birr 258 and sold it at Birr 288, find
   a. markup
   b. markup percent with respect to
4. Ato Dechassa wants to sell his ox at Birr 3,652 with a 10% markup on his cost. Find the cost of the ox.
5. Martha bought a shoe for Birr 280 and wants to sell it at 24% markup. Find
   a. markup
   b. selling price of the shoe
6. Abebe sold a quintal of Teff at Birr 1,068 with 45% markup on selling price. Find the cost.
7. Find the percent markup on cost, if mark - up on selling price is 30%.

11.2 COMPOUND INTEREST AND DEPRECIATION

ACTIVITY 11.4



Suppose you deposit Birr 100 in a bank.

The bank calculates interest for you at a rate of 4% per year compounded semi-annually. What is your amount of money at the end of 2 years?

Simple Interest

When money is borrowed, or you deposit money in an account, a fee is paid for the use of the money. A fee paid for the use of money is called **interest**. From the investment point of view, interest is income from invested capital. The capital originally invested is called the **principal (or present value).** The sum of the principal and interest due (or paid) is called the **amount (or future value or accumulated value)**.

For simple interest, the interest is computed on the original principal during the whole time, or term of the loan; at the stated annual rate of interest. The computation of simple interest is based on the following formula:

Simple interest: $I = Prt$

Where I is the simple interest, P is the principal, r is the interest rate per year or annual interest rate, and t is the time in years.

Note:

The time period for r and t must be consistent with each other. That is, if r is expressed as percentage per year, then t should be expressed in number of years.

In general, if a principal P is borrowed at a rate r of simple interest per year for t years, then the borrower will pay back the lender an amount A that include the principal plus the amount of interest I.

$$A = P + I = P + Prt = P(1 + rt)$$

Therefore, to compute the future value of a simple interest, we use the formula:

The future value of a simple interest:

$$A = P(1 + rt)$$

where A is the future value, P is the principal, r is the simple interest rate per year, and t is the time in years.

Example 1 If Birr 2,500 is invested with a simple interest rate of 2% per month, find the amount of the interest and future value at the end of the fourth month.

Solution In this example you have the principal P = Birr 2,500, the interest rate per month $r = 0.02$, and the time $t = 4$ months.

$I = Prt$ where r is the interest rate per periods that is interest rate per month.

$I = Prt = 2500 \times 0.02 \times 4 =$ Birr 200.

The value of the investment after four months is

$A = P + I = 2{,}500 + 200 =$ Birr 2,700.

Example 2 Zenebech wants to buy an electric stove priced at Birr 2,500 and agreed to pay Birr 700 initially and the remaining amount to be equally paid monthly on simple interest rate of 13% per year in 9 months (i.e. the remaining amount plus its interest). What is the monthly payment she has to do?

Solution The amount of loan = Birr 2,500 – Birr 700 = Birr 1,800.

Hence, the Principal will be 1,800 Birr,

interest rate $r = 0.13$, time $t = \frac{9}{12}$ years and

the number of times payment is made is $n = m \times t = 12 \times \frac{9}{12} = 9$ times,

where m is the number of times payment is made per year.

Therefore, the periodic payment is

$$\text{Periodic payment} = \frac{P + I}{n} = \frac{P(1 + rt)}{mt} = \frac{1800\left[1 + 0.13\left(\frac{9}{12}\right)\right]}{12 \times \frac{9}{12}}$$

$$= \frac{1975.5}{9} = \text{Birr } 219.5$$

11.2.1 Compound Interest

If at the end of a payment period the interest due is reinvested at the same rate, the interest as well as the original principal will earn interest during the next payment period. Interest paid on interest reinvested is called **compound interest**.

If P is the principal earning interest compounded annually at a rate of r per year for n years, then the amount at the end of one year can be calculated from the simple interest relation

$A = P(1 + rt)$

The amount at the end of the first year A_1 (i.e., when $t = 1$) is

$$A_1 = P(1+r)$$

Since the amount at the end of the first year will serve as principal for the second year, at the end of the second year the amount A_2 will be

$$A_2 = A_1(1+r) = P(1+r)(1+r) = P(1+r)^2 .$$

Since the amount at the end of second year will serve as principal for the third year, at the end of the third year the amount A_3 will be

$$A_3 = A_2(1+r) = P(1+r)^2(1+r) = P(1+r)^3 .$$

Similarly, since the amount at the end of the third year will serve as principal for the fourth year, at the end of the fourth year the amount A_4 will be;

$$A_4 = A_3(1 + r) = P(1 + r)^3 (1 + r) = P(1 + r)^4$$

Continuing this process, we see that the amount at the end of the n^{th} year will be

$$A_n = A_{n-1}(1+r) = P(1+r)^{n-1}(1+r) = P(1+r)^n$$

Therefore, the total amount A after n years will be given by

$$A = P(1+r)^n \cdots\cdots\cdots (*)$$

Interest is usually compounded more than once a year. The quoted rate of interest per year is called **annual** or **nominal rate** and the interval of time between successive interest calculations is called **conversion period** or **compound period**.

Example 3 Find the amount of interest on a deposit of Birr 1,000 in an account compounded annually with annual interest rate of 6% for 5 years.

Solution We are given P = Birr 1,000, $r = 0.06$, $t = 5$ years and we need to find the future value A and then the amount of interest.

$A = P(1 + r)^n = 1{,}000(1.06)^5$ = Birr 1,338.23.

Hence the amount of the compound interest of the deposit is

$I = A - P$ = 1,338.23 – 1,000.00 = Birr 3,38.23.

If interest at an annual rate of r per year is compounded m times a year on a principal P, then the simple interest rate per conversion period is

$$i = \frac{\text{annual interest rate}}{\text{number of periods per year}} = \frac{r}{m}$$

Since r is the annual interest rate and the interest is compounded m times per year, the year is divided into m equal conversion periods and the interest rate during each conversion period is $i = \frac{r}{m}$; that is, we get interest $\frac{r}{m}$ every $\frac{1}{m}$ years.

Now, if the interest is compounded for t years, then there will be $n = mt$ conversion periods in t years. Thus if you put $n = mt$ and replace r by the expression of interest rate per each conversion period $i = \frac{r}{m}$ in equation $(*)$, we have the future value of compound interest given by;

Future value of a compound interest:

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

where A is amount or future value, P is principal or present value, r is annual or nominal rate, t is time in years, and m is the number of conversion periods per year.

In working with problems involving interest, we use the term of payment periods as follows:

- ✓ Annually means once a year, i. e. $m = 1$
- ✓ Semi-annually means twice a year, i. e. $m = 2$
- ✓ Quarterly means four times a year, i. e. $m = 4$ and
- ✓ Monthly means 12 times a year, i. e. $m = 12$.

Now, study the following examples to understand the concepts you have discussed above.

Example 4 If Birr 100 is deposited in the Commercial Bank of Ethiopia with interest rate of 10% per annum, find the amount if it is compounded annually, semi-annually, quarterly, monthly, and weekly at the end of one year. (No withdrawal or deposit is made in the whole year).

Solution You are given the principal P = Birr 100, the annual interest rate $r = 0.1$, for a period of time $t = 1$ year, and compound period of $n = mt$:

a Annually means $m = 1$, so that the amount at the end of the year is

$$A = P\left(1 + \frac{r}{m}\right)^{mt} = 100\left(1 + \frac{0.1}{1}\right)^{1(1)} = \text{Birr } 110.$$

b Semi-annually means $m = 2$, so that the amount at the end of the year is

$$A = P\left(1 + \frac{r}{m}\right)^{mt} = 100\left(1 + \frac{0.1}{2}\right)^{2(1)} = \text{Birr } 110.25.$$

c Quarterly means $m = 4$, so that the amount at the end of the year is

$$A = P\left(1 + \frac{r}{m}\right)^{mt} = 100\left(1 + \frac{0.1}{4}\right)^{4(1)} = \text{ Birr } 110.38.$$

d Monthly means $m = 12$, so that the amount at the end of the year is

$$A = P\left(1 + \frac{r}{m}\right)^{mt} = 100\left(1 + \frac{0.1}{12}\right)^{12(1)} = \text{ Birr } 110.47.$$

e Weekly means $m = 52$, so that the amount at the end of the year is

$$A = P\left(1 + \frac{r}{m}\right)^{mt} = 100\left(1 + \frac{0.1}{52}\right)^{52(1)} = \text{ Birr } 110.51.$$

We can summarize the above result in the table given below.

	Number of times interest is compounded	Amount at the end of one year
Annually	1	Birr 110.00
Semi-annually	2	Birr 110.25
Quarterly	4	Birr 110.38
Monthly	12	Birr 110.47
Weekly	52	Birr 110.51

Interest compounded at different time periods in one year

You can observe that when the time, principal and interest rates are kept fixed and the number of times the interest is compounded increases, the amount will increase.

Example 5 Suppose Birr 2,300 is invested at 8% interest rate compounded

a annually **b** monthly.

What is the amount after 5 years? Find the amount of interest in each case.

Solution Given P = Birr 2,300, $r = 0.08$, and $t = 5$ years.

a When the interest is compounded annually, $m = 1$ and hence the amount will be

$$A = P\left(1 + \frac{r}{m}\right)^{mt} = 2,300\left(1 + \frac{0.08}{1}\right)^{1(5)} = 2,300(1.469328) = \text{ Birr } 3,379.45.$$

The interest earned in five years without making withdrawal or deposit will be

$I = A - P = 3{,}379.45 - 2{,}300 =$ Birr 1,079.45.

b When the interest is compounded monthly, $m = 12$ and hence the amount will be

$$A = P\left(1+\frac{r}{m}\right)^{mt} = 2,300\left(1+\frac{0.08}{12}\right)^{12(5)} = 2,300\left(1+\frac{0.08}{12}\right)^{60} = 2,300(1.00667)^{60}$$

$= \text{Birr } 3,427.33$

The interest earned in five years without making withdrawal or deposit will be

$I = A - P = 3,427.33 - 2300 = \text{Birr } 1,127.33.$

When people engaged in finance speak of the "time value of money", they are usually referring to the present value of money. The present value of Birr A to be received at a future date is the principal you would need to invest now so that it would grow to Birr A in the specified time period. From the future value of a compound investment, you can get a formula for the present value. If P is the present value of Birr A to be received after t years at annual interest rate r compounded m times a year, then $A = P\left(1+\frac{r}{m}\right)^{mt}$

To solve for P, divide both sides by $\left(1+\frac{r}{m}\right)^{mt}$, and we obtain the present value of a compound interest expressed as:

$$P = A\left(1+\frac{r}{m}\right)^{-mt}$$

Example 6 Find the present value of an investment that will grow to Birr 600 after two years compounded quarterly at the interest rate of 9% per year.

Solution The given information is A = Birr 600, $t = 2$ years, $m = 4$, and $r = 0.09$. We want to find the present value P. The present value is given by

$$P = A\left(1+\frac{r}{m}\right)^{-mt} = 600\left(1+\frac{0.09}{4}\right)^{-4(2)} = 600(1.0225)^{-8} = \text{ Birr } 502.16.$$

Example 7 Ato Mohammed made the following transactions on his account at the Commercial Bank of Ethiopia. Deposited Birr 2500 on 1st January 2006; withdraw Birr 600 on 1st July 2007; deposited Birr 1,800 on 1st January 2008. If the account earns 4% interest rate per year compounded semi-annually, find the balance on the account on 1st January 2010.

Solution From the 1st January 2006 up to 1st July 2007 we have 18 months which is 3 conversion periods. Hence we are given P = Birr 2,500, $t = 1.5$ years, $m = 2$, and $r = 0.04$. Hence the amount will be:

$$A = P\left(1+\frac{r}{m}\right)^{mt} = 2,500\left(1+\frac{0.04}{2}\right)^{2\left(\frac{3}{2}\right)} = \text{Birr } 2,653.02.$$

The balance on 1st July 2007 will be 2653.02 Birr. If a withdrawal of Birr 600 is made on this day, the balance will be Birr 2,653.02 – Birr 600.00 = 2,053.02 Birr.

From the 1st July 2007 up to 1st January 2008 we have 6 months which is 1 conversion period. Hence we are given P = Birr 2,053.02, $t = 0.5$ years, $m = 2$, and $r = 0.04$. Hence the amount will be:

$$A = P\left(1+\frac{r}{m}\right)^{mt} = 2,053.02\left(1+\frac{0.04}{2}\right)^{2\left(\frac{1}{2}\right)} = \text{Birr } 2,094.08.$$

Since he made a deposit of Birr 1,800 on this day, the balance on 1st January 2008 will be Birr 2094.08 + Birr 1,800.00 = Birr 3,894.08.

From 1st January 2008 up to January 2010 we have 2 years, that is four conversion periods. Hence we are given P = Birr 3,894.08, $t = 2$ years, $m = 2$, and $r = 0.04$. Hence the amount will be:

$$A = P\left(1+\frac{r}{m}\right)^{mt} = 3,894.08\left(1+\frac{0.04}{2}\right)^{2(2)} = \text{Birr } 4,215.08.$$

Thus, the balance on 1st of January 2010 will be Birr 4,215.08.

Ordinary annuity

Many people are not in a position to deposit a large amount of money at a time in an account. Most people save money by depositing relatively small amount at different times. If a depositor makes equal deposits at regular intervals, he/she is contributing to an **annuity**. The deposits may be made weekly, monthly, yearly, or any other period of time.

If we deal with annuities in which the deposits (or payment) are made at the end of each of the deposit (or payment) intervals, which coincides with the compounding period of interest, then this type of annuity is called **ordinary annuity**. In this section we will deal with future value of an ordinary annuity only and start the discussion with the following example.

Example 8 Suppose you deposit Birr 100 at the end of every six months in an account that pays 4% interest per year compounded semi-annually. If you made 8 deposits, one at the end of each interest payment period over four years, how much money will you have in the account at the end of the four years?

Solution If you make the payments at the end of each six months, coinciding with the time interest is compounded, you start the discount from the last payment.

The eighth payment has no interest, so stays at Birr 100.

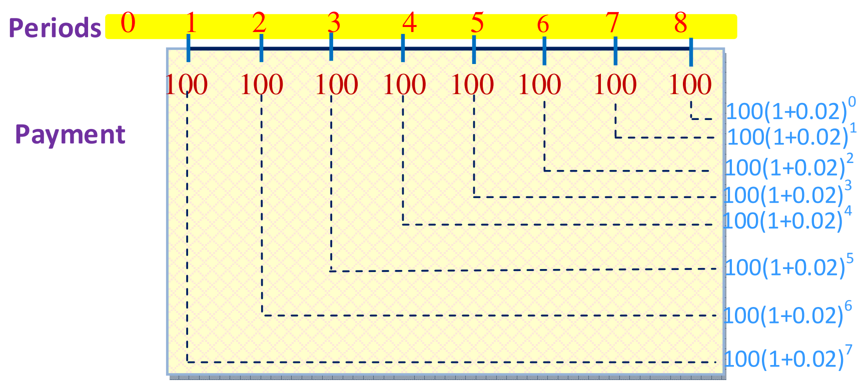
The seventh payment has interest calculated for one period, and it will accumulate to $A = P(1 + it)$, where $P = 100$ periodic payment, $i = \frac{r}{m} = \frac{0.04}{2} = 0.02$ is the interest rate per period, and $t = 1$ period.

Therefore $A = P(1 + it) = 100(1 + 0.02(1)) = 100(1 + 0.02)$

The sixth payment has interest commutated for two periods, and it will accumulate for the first period $A = P(1 + it) = 100(1 + 0.02(1)) = 100(1 + 0.02)$, and for the second period as the amount for the first period serve as a principal to the second period $A = P(1 + it) = 100(1 + 0.02)(1 + 0.02(1)) = 100(1 + 0.02)^2$

The fifth payment has interest computed for three periods, and it will accumulate to the amount $A = 100(1 + 0.02)^3$.

Continuing this process the first payment has interest computed for seven periods, and will accumulate to the amount $A = 100(1 + 0.02)^7$ as illustrated in the following diagram.



The amount of the ordinary annuity will be the sum S of the amount accumulated from each deposit made, that is,

$$S = 100 + 100(1 + 0.02) + 100(1 + 0.02)^2 + 100(1 + 0.02)^3 + \ldots + 100(1 + 0.02)^7$$
$$= 100 + 100(1.02) + 100(1.02)^2 + 100(1.02)^3 + \ldots + 100(1.02)^7$$

To find the sum S, multiply S by 1.02 and subtract S from it term by term.

$$1.02S = 100(1.02) + 100(1.02)^2 + 100(1.02)^3 + 100(1.02)^4 + \ldots + 100(1.02)^8$$
$$\underline{S = 100 + 100(1.02) + 100(1.02)^2 + 100(1.02)^3 + \ldots + 100(1.02)^7}$$
$$0.02S = 100(1.02)^8 - 100 \Rightarrow 0.02S = 100((1.02)^8 - 1)$$

Therefore, we have $S = 100\left(\frac{(1.02)^8 - 1}{0.02}\right)$ = Birr 858.29. (Using a calculator)

In general, to determine the sum S that a series of deposits of R will grow after n periods, we have

$$\begin{aligned}(1+i)S &= R(1+i)+R(1+i)^2+R(1+i)^3+R(1+i)^4+\ldots+R(1+i)^n\\ S &= R+R(1+i)+R(1+i)^2+R(1+i)^3+\ldots+R(1+i)^{n-1}\\ \hline iS &= R(1+i)^n - R\\ iS &= R\left((1+i)^n-1\right)\end{aligned}$$

Therefore, we have $S = R\left(\frac{(1+i)^n-1}{i}\right)$.

The future value of an ordinary annuity is given by

$$S = R\left(\frac{(1+i)^n-1}{i}\right)$$

where R is the periodic payment, i is the interest rate per period and n is the number of periods.

✍Note:

The amount of interest of an ordinary annuity is $I = S - nR$.

$i = \frac{r}{m}$ and $n = mt$, in which r is the interest rate per year, m is the number of times interest is compounded per year and t is time in years.

Example 9 Elizabeth deposits Birr 350 at the end of every month into a saving account that pays an interest rate of 12% per year compounded monthly. How much money is in her account at the end of 5 years? What is the amount of interest?

Solution You are given R = Birr 350, $r = 0.12$, m = 12, and $t = 5$ years. To use the above formula we need to find $i = \frac{r}{m} = \frac{0.12}{12} = 0.01$ and $n = mt = 12(5) = 60$.

a The accumulated balance is given by

$$S = R\left(\frac{(1+i)^n-1}{i}\right) = 350\left(\frac{(1+0.01)^{60}-1}{0.01}\right) = \text{Birr } 28{,}584.38.$$

b The amount of interest is $I = S - nR = 28{,}584.38 - 60(350) =$ Birr 7,584.38.

Exercise 11.5

1 If Ato Abebe deposits a sum of money in a bank at 7.5 % interest rate per year compounded monthly, then how long will it take to double?

2 Ato Lemma works in XYZ-company earning a monthly salary of Birr 2,400. He is also a member of the credit association of his company and deposits 20% of his monthly salary at the end of each month at 4% compounded monthly.

   a What is Ato Lemma's accumulated balance by the end of three years?

   b How much interest has he earned?

3 If Dalelo deposited Birr 1,000 saving at 7% interest per year, how much will the amount be at the end of 10^{th} year?

4 Helen deposited Birr 2,000 at 8% interest compound annually. How many years will it take her to get Birr 3,000?

5 Suppose you deposit Birr 100 in an account at the end of every quarter with 8% interest compounded quarterly. How much amount will you have at the end of 5 years?

6 An amount of Birr 500 is deposited in an account at the end of each six-month period with an interest computed at 6% compounded semi-annually. How many years does it take for the amount to reach Birr 56,398.43?

11.2.2 Depreciation

Any physical thing (tangible) or right (intangible such as, patents, copyrights and goodwill) that has money value is an **asset**. There are two groups of assets known as **current assets** (**financial assets**) and **plant assets** (or **fixed assets**).

Cash and other assets that may reasonably be expected to be recognized in cash or sold or consumed within one year or less through the normal operation of the business are called **current assets**.

Tangible assets used in business (not held for sales in the ordinary course of the business) that are of a permanent or relatively fixed nature are called **plant assets or fixed assets**.

Suppose a photographic equipment is used in the operation of a business. It is obvious that the equipment does wear out with usage and that its usefulness decreases with the passage of time. The decrease in usefulness is a business expense, called **depreciation**. Plant assets include equipment, machinery, building, and land. With the exception of land, such assets gradually wear out or otherwise lose their usefulness with passage of time, i.e. they are said to depreciate. Since we are interested in this subsection how plant assets depreciate, from now on you consider plant assets to be simply assets.

The depreciation of an asset is caused mainly due to:

- **a** **physical depreciation:-** wear out from use and deterioration from the action of the element
- **b** **functional depreciation:-** inadequacy and obsolescence. Inadequacy results if the capacity does not meet the demand of increased production, while obsolescence results, if the commodity produced is no longer in demand with respect to quality and cost of production.

Factors to be considered in computing the periodic depreciation of an asset are its original cost, its recoverable cost at the time it is retired from service, and the length of life of the asset. It is evident that neither of these two latter factors can be accurately determined until the asset is retired; they must be estimated at the time the asset is placed in service. The estimated recoverable cost of depreciable asset as of the time of its removal from service is variously termed as **residual value**, **scrap value**, **salvage value**, or **trade-in value**.

There is no single method of computing depreciation for all classes of depreciable assets. Here we consider two methods:

- **i** The fixed instalment method and
- **ii** Reducing-balance method

The fixed instalment method

The fixed instalment method (or on cost method or the straight-line method) of determining depreciation allows for equal periodic charges to expense (or cost) over the estimated life of the asset. That is, under this method, the depreciation is charged evenly every year throughout the economic life of the asset. The periodic depreciation charge of an asset is expressed as:

$$\text{depreciation} = \frac{\text{cost} - \text{salvage value}}{\text{estimated life in years}}$$

This method is quite simple to apply as the arithmetical calculations are very easy but there are certain disadvantages of this method:

- **i** The method does not take into consideration the seasonal fluctuations, booms and depreciation.
- **ii** The usefulness of machinery is more in earlier years than it is in later years.
- **iii** The total charges in respect of an asset are not equal every year because repairs are much less in earlier years.

Example 10 A machine costing Birr 35,000 is estimated to have a useful lifetime of 8 years and a salvage value of Birr 3,000. What is the accumulated depreciation at the end of 5 years? Find the book value of the asset at that time, using the fixed instalment method

(where book value = cost – accumulated depreciation)

Solution We have the cost = Birr 35,000, salvage value = Birr 3,000 and useful life = 8 years.

The depreciation charge per year is

$$\text{depreciation} = \frac{\text{cost} - \text{salvage value}}{\text{estimated life in years}} = \frac{35,000 - 3,000}{8} = \text{Birr } 4,000$$

Hence the accumulated depreciation increases by Birr 4,000 every year.

The accumulated depreciation at the end of 5 years will be:

years × depreciation charge per year = 5×(4,000) = Birr 20,000.

The book value of the asset at the end of 5 years will be:

Book value = cost – accumulated depreciation = 35,000 – 20,000 = Birr 15,000.

The depreciation schedule for the asset is shown in the following table.

Number of years	Yearly depreciation	Accumulated depreciation	Book value
0	0	0	35,000
1	4,000	4000	31,000
2	4,000	8,000	27,000
3	4,000	12,000	23,000
4	4,000	16,000	19,000
5	4,000	20,000	15,000
6	4,000	24,000	11,000
7	4,000	28,000	7,000
8	4,000	32,000	3,000

Example 11 Office furniture was purchased on September 18 for Birr 2,020. The salvage value of the furniture is Birr 250, and the estimated life is 10 years. What is the book value at the end of the fourth year using the fixed installment method?

Solution Note that a calendar month is the smallest unit of time employed to estimate the life of an asset. When this time interval is adopted, all assets placed in service or retired from service during the first half of a month are treated as if the event has occurred on the first day of that month.

Similarly, all plant assets (additions or reductions) during the second half-month are considered to have occurred on the first day of the next month.

Since the date of purchase is on September 18, it is close to October 1. The depreciation for the first month is based on October 1. The depreciation charge per year is

$$\text{depreciation} = \frac{\text{cost} - \text{salvage value}}{\text{estimated life in years}} = \frac{2020-250}{10} = \text{Birr } 177 \text{ per year.}$$

From the yearly depreciation of Birr 177, we can find the monthly depreciation by dividing it by 12 as follows.

Birr 177 per year $\div$ 12 = Birr 14.75 per month.

Since from October 1 through the end of the year, December 31, encompasses 3 months, we multiply the monthly depreciation by 3 to get the depreciation for the first year as Birr 14.75 per month $\times$ 3month = Birr 44.25.

From the second year through the tenth year, the full Birr 177 per year is taken as depreciation. Hence the depreciation at the end of the fourth year will be

44.25 + 3(177) = Birr 575.25

Hence the book value at the end of the fourth year will be:

book value = cost – depreciation = 2020 − 575.25 = Birr 1444.75.

Reducing balance method

The reducing balance method (or declining-balance method) yields a declining periodic depreciation charge over the estimated life of the asset. Of the several variant techniques the most common is to apply double straight-line depreciation rate, computed by:

$$\textit{Annual percentage rate of depreciation} = 2\times\frac{100\%}{\textit{Estimated life time}} = \frac{200\%}{\textit{Estimated life time}}$$

The double reducing balance method uses the double rate applied to the cost of the asset for the first year of its use and thereafter to the declining book value at the beginning of the year, i.e. cost minus the accumulated depreciation.

Example 12 A company machine is purchased for Birr 3,217.89. The expected life is 4 years. Use double reducing balance method to prepare a depreciation schedule.

Solution The annual percentage rate of depreciation is

$$\frac{200\%}{\text{estimated life time}} = \frac{200\%}{4} = 50\%$$

The yearly depreciation and book value are shown in the following table.

Year	Book value at the beginning of the year	Rate	Depreciation calculation	Depreciation for the year	Accumulated Depreciation	Book value at the end of the year
1	3217.89	0.5	3217.89×0.5	1608.95	1608.94	1608.94
2	1608.94	0.5	1608.94×0.5	804.47	2413.41	804.47
3	804.47	0.5	804.48×0.5	402.24	2815.65	402.24
4	402.24	0.5	402.24×0.5	201.12	3.016.77	201.12

Example 13 Using the double reducing balance method of depreciation, determine the book value at the end of the second year of an item that was bought on May 5 for Birr 30,000 and that has a salvage value of Birr 5,000 and an estimated useful life of 40 years.

Solution The depreciation rate per year is $\frac{200\%}{\text{Estimated life time}} = \frac{200\%}{40} = 0.05$

The depreciation for the first full year is $30{,}000\times0.05=$ Birr 1,500.

Hence the depreciation per month is

Birr 1,500 per year $\div$12 month per year = Birr 125 per month.

Since the item is bought on May 5, it is close to May 1. Hence at the end of the first year the depreciation is

Birr 125 per month $\times$8 months = Birr 1000.

The book value at the end of the first year is 30,000 – 1000 = Birr 29,000.

Therefore, the depreciation for the second year is

29,000 $\times$0.05 = Birr 1450, and

the book value at the end of the second year is

Birr 29,000 – Birr 1,450 = Birr 27,550.

Exercise 11.6

New equipment was obtained at a cost of Birr 100,000 on January 5. The equipment has estimated lifetime of 5 years and an estimated residual value of Birr 8,000.

i Determine the annual depreciation for each of the five years of the estimated useful life of the equipment.

ii The accumulated depreciation at the end of each year.

iii The book value of the equipment at the end of each year by using

a the fixed instalment method.

b the double reducing balance method.

11.3 SAVING, INVESTING AND BORROWING MONEY

Group Work 11.1

1 Who makes most decisions about how much to save and invest in a market economy, and about how to save and invest?

2 Why are banks and financial markets important to economic growth?

3 Why are individuals in households and businesses more likely to make saving and investment decisions that advance their own economic interests more effectively than decisions made by government officials?

What is Money?

It is very difficult to give a precise definition of money because various authors have defined money differently. However we may define money in terms of functions it performs, i.e. "Money is that what money does" or "Anything which is generally accepted as a medium of exchange in the settlement of all transactions including debt and acts as a measure and store of value".

ACTIVITY 11.5

Give reasons to make a Birr money?

Functions of money

Money performs the following four important functions

a **Money as a medium of exchange:** the most important function of money is to serve as a medium of exchange.

b **Money as a measure of value:** money serves as a common measure of value or unit of account. It serves as a standard or yardstick in terms of which values of all goods and services can be expressed.

c **Money as a standard of deferred payment:** money serves as a standard in terms of which future payments can be expressed.

d **Money as a store of value:** money being the most liquid of all assets is a convenient form in which to store wealth. Furthermore, money helps in the transfer of value from one person to another as well as from one place to another.

The first two functions are called **primary functions of money** and the last two are called **secondary functions of money**.

11.3.1 Saving money

A Reasons for saving

You may be asking yourself why there is so much pressure to save money. If you have enough to pay for everything you need, why should you worry about putting any aside each month? There are a variety of reasons to begin saving money. Different people save for different reasons. Here are seven reasons that you may consider for saving your money.

1. **Save for emergency funds**
2. **Save for retirement**
3. **Save for a down payment on a house**
4. **Save for vacations and other luxury items**
5. **Save for a new car**
6. **Save for sinking funds**
7. **Save for your education**

Group Work 11.2

Form a group and study the following issues.

Consider the family of each member in your group and let each student ask his/her family.

- **i** Whether they save money or not.
- **ii** If yes, why do they save?

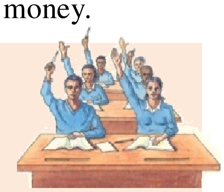
After collecting this data discuss

- **a** the seven reasons mentioned on why we save money,
- **b** your findings with respect to the above reasons of saving.

B Planning a saving programme

If you think yourself as an employee or a business man, you need to plan on how to save, and this planning is directly related to the reason on why you save money.

ACTIVITY 11.6

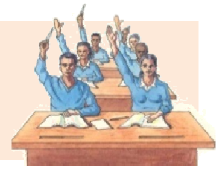


If you are a government employee, discuss a plan on how you should save for:

- **a** retirement,
- **b** vacations,
- **c** a down payment on a house.

C Savings as investment

ACTIVITY 11.7



Discuss how you should plan to save and be involved in an investment.

New issues of corporate stock: New corporations raising funds to begin operation, or existing corporations that want to expand their current operations, can issue new shares of stock through the investment banking process. People who buy these shares of stock hope to make money by having the price of the stock increase, and through dividends that may be paid out of future profits.

New issues of bonds: New issues of bonds are issued by companies that want to borrow funds to expand by investing in new factories, machinery, or other projects, and by government agencies that want to finance new building, roads, schools, or other projects. The bonds are promises to repay the amount borrowed, plus interest, at specified times.

Individuals, banks, or companies that want to earn this interest purchase the bonds.

Borrowing from banks and other financial intermediaries: Companies (and individuals) can borrow funds from banks, agreeing to pay interest, on a specified schedule. Banks and other financial intermediaries lend out money that has been deposited by other people and firms. In effect, banks and other intermediaries are just a special kind of "middleman," making it easier for those with money to lend to find those who want to borrow funds. Of course, banks also screen those who borrow money, to make sure they are likely to repay the loans.

D Saving institutions

Group Work 11.3

Form a group and discuss the following.

1 What are saving institutions?

2 Is there any saving institution in your surrounding?

3 Visit any saving institution in your surrounding and study how it works.

4 Present your findings to the class.

Saving institutions are financial institutions that raise loanable funds by selling deposits to the public. They accept deposits from individuals and firms and use these funds to participate in the debt market, making loans or purchasing other debt instruments such as Treasury bills. The major types of saving financial institutions are commercial banks, saving and loan associations, mutual saving banks, and credit unions. Their main liabilities (sources of funds) are deposits, and their main assets are loans.

i Commercial banks

Commercial banks are business corporations that accept deposits, make loans, and sell other financial services, especially to other business firms, but also to households and governments.

ii Savings and loans associations

Savings and loans associations (S & Ls) were originally designed as mutual associations, (i.e., owned by depositors) to convert funds from savings accounts into mortgage loans.

iii Mutual savings banks

Mutual savings banks are much like savings and loans, but are owned cooperatively by members with a common interest, such as company employees, union members, or congregation members.

iv Credit unions

Credit unions are non-profit associations accepting deposits from and making loans to their members, all of whom have a common bond, such as working for the same employer. Credit unions are organized as cooperative depository institutions, much like mutual savings banks. Depositors are credited with purchasing shares in the cooperative, which they own and operate.

Exercise 11.7

What type of financial institutions would each of the following people be most likely to do business with

- **a** a person with Birr 10,000 in savings who would like to earn a decent return at low risk and who does not know much about the stock and bond markets,
- **b** a person with Biir 350 who needs a checking account,
- **c** a person who needs a Birr 10,000 loan to open a pizza shop
- **d** a person who is recently married, is starting a family, and wants to make sure that his children are well taken care of in the future,
- **e** the president of a small company who wants to list it on the stock exchange to obtain additional capital,
- **f** someone who has just received a large inheritance and wants to invest it in the stock market,
- **g** a person with no credit history who is buying her first car,
- **h** a family needing a mortage loan to buy a house,
- **i** a person who has declared bankruptcy in the past and is looking for a loan to pay off some past due bills.

11.3.2 Investment

Investment is the production and purchase of capital goods, such as machines, building, and equipment that can be used to produce more goods and services in the future. Personal investment is purchasing financial securities such as stocks and bonds, which are riskier than savings accounts because they may fall in value, but in most cases, will pay a high rate of return in the long run than the interest paid on savings accounts.

Group Work 11.4

1. What is an investment.
2. Discuss any investment activities in your surrounding.
3. Discuss any relation between the financial institutions and the investment(s) in your surrounding.

A Investment strategy

In finance, an investment strategy is a set of rules, behaviours or procedures, designed to guide an investor's selection of an investment portfolio. Usually the strategy will be designed around the investor's risk-return tradeoff. Some investors will prefer to maximize expected returns by investing in risky assets, others will prefer to minimize risk, but most will select a strategy somewhere in between.

Passive strategies are often used to minimize transaction costs, and active strategies such as market timing are an attempt to maximize returns. One of the better known investment strategies is buy and hold. Buy and hold is a long term investment strategy, based on the concept that in the long run equity markets give a good rate of return despite periods of volatility or decline.

B Types of securities

Stocks

Stocks can help you build long-term growth into your overall financial plan. History has repeatedly demonstrated that stocks, as an asset class, have outperformed every other type of investment over long periods of time. Stock represents an ownership or equity stake in a corporation. If you are a stockholder, you own a proportionate share in the corporation's assets and you may be paid a share of the company's earnings in the form of dividends.

Stocks are considered to be a riskier investment than bonds or cash. Stock prices tend to fluctuate more sharply-both up and down than other types of asset classes.

ACTIVITY 11.8

1 After reading literatures of financial securities, state at least four of the main characteristics that may distinguish preferred stock from common stock.

2 After reading additional financial security books, state at least four basic rights that can come from ownership of stock in a corporation.

Bonds

Corporations, governments and municipalities issue bonds to raise funds, and in return they typically pay the bond owners a fixed interest rate. In this way, a bond is like a loan.

Bonds may provide a regular income stream or diversify a portfolio. Bonds are fixed income investments - most pay periodic interest and principal at maturity.

Interest rates may be the most significant factors affecting a bond's value. When interest rates fall, the value of existing bonds rise because their fixed-interest rates are more attractive in the market than the rates for new issues. Similarly, when interest rates rise, the value of existing bonds with lower, fixed-interest rates tend to fall.

Inflation may erode the purchasing power of interest income. Generally, bonds with longer maturities are more sensitive to inflation than bonds with shorter maturities. Economic conditions may cause bond values - particularly corporate bonds-to fluctuate. An economic change that adversely affects a company’s business may reduce the ability of a company to make interest or principal payments.

ACTIVITY 11.9

After reading literatures of financial securities, state the difference between preferred stock and bonds.

C How to invest

As you may have noticed, there are several categories of investments, and many of those categories have thousands of choices within them. So finding the right ones for you isn't a trivial matter. The single greatest factor, by far, in growing your long-term wealth is the rate of return you get on your investment. There are times, though, when you may need to park your money someplace for a short time, even though you won't get very good returns. Here is a summary of the most common short-term savings vehicles:

Short-term savings vehicles

Savings account: Often the first banking product people use, saving accounts, earn a small amount in interest, so they're a little better than that dusty piggy bank on the dresser.

Money market funds: These are a specialized type of mutual fund that invests in extremely short-term bonds. Money market funds usually pay better interest rates than a conventional savings account does, but you'll earn less than what you could get in certificates of deposit.

Certificate of deposit (CD): This is a specialized deposit you make at a bank or other financial institution. The interest rate on certificate of deposits is usually about the same as that of short- or intermediate-term bonds, depending on the duration of the CD. Interest is paid at regular intervals until the certificate of deposit matures, at which point you get the money you originally deposited plus the accumulated interest payments. Fools are partial to investing in stocks, as opposed to other long-term investing vehicles, because stocks have historically offered the highest return on our money. Here are the most common long-term investing vehicles:

Long-term investing vehicles

Bonds: Bonds come in various forms. They are known as "fixed-income" securities because the amount of income the bond generates each year is "fixed" or set, when the bond is sold. From an investor's point of view, bonds are similar to CDs, except that the government or corporations issue them, instead of banks.

Stocks: Stocks are a way for individuals to own parts of businesses. A share of stock represents a proportional share of ownership in a company. As the value of the company changes, the value of the share in that company rises and falls.

Mutual funds: Mutual funds are a means for investors to pool their money to buy stocks, bonds, or anything else the fund manager decides is worthwhile. Instead of managing your money yourself, you turn over the responsibility of managing that money to a professional. Unfortunately, the vast majority of such "professionals" tend to under-perform the market indexes.

Exercise 11.8

Direction:- *Mark an S if the situation involves saving, an I, if the situation involves investing, a P if the situation involves personal investing, and an N if the situation involves neither saving nor investing.*

a Kassech borrowed Birr 25,000 from a bank to purchase a computer and other equipment and supplies to open her new internet home page business.

b Bontu buys 100 shares of Alpha PLC, hoping that the price per share will increase.

c Mike dies and leaves his estate of Birr100, 000 to his five children. They use it to take an around-the-world, once-in-a-lifetime, one-year cruise.

d Dawit, the head of Sunshine Computer Systems, issues new shares of stock in his company through an investment banker, and uses those funds to build a new assembly line to produce the world's fastest microprocessors.

e A woman takes a new job and has Birr 20 a week deducted from each paycheck to be deposited directly into a savings account at her bank.

f Ford Motor Company issues a Birr 5,000 bond, which is purchased by Sara.

g Medical Systems, Inc. builds a new plant to produce experimental pacemakers.

h Mark quits his job to go back to school to study economics, hoping to earn more money with a college degree.

11.3.3 Borrowing Money

Group Work 11.5

Discuss:

a how one borrows money.

b from where one can borrow money.

c institutions that give loans.

d why we borrow money.

e the advantages and disadvantages of borrowing money.

Loans, overdrafts and buying on credit are all ways of borrowing. Different methods of borrowing suit different types of people and situations. Whatever type of borrowing you choose, it is important to make sure you will be able to afford the repayments.

Types of loan

Secured loan

With a secured loan, the lender has the right to force the sale of the asset against which the loan is secured if you fail to keep up the repayments. The most common form of secured loan is called a 'further advance' and is made against your home by borrowing extra on your mortgage. (Your mortgage is itself a secured loan.) Because secured loans are less risky for the lender, they are usually cheaper than unsecured loans. Secured loans are mostly suitable for borrowing large amounts of money over a longer term, for example, for home improvements.

Unsecured loan

An unsecured loan means the lender relies on your promise to pay it back. They're taking a bigger risk than with a secured loan, so interest rates for unsecured loans tend to be higher. Unsecured loans are often more expensive and less flexible than secured loans, but suitable if you want a short-term loan (one to five years).

Credit union loan

Credit Unions are mutual financial organizations which are owned and run by their members for their members. Once you've established a record as a reliable saver they will also lend you money but only what they know you can afford to repay. Members have a common bond, such as living in the same area, a common workplace, membership of a housing association or similar.

Money lines

Money lines are community development finance institutions that lend and invest in deprived areas and underserved markets that cannot access mainstream finance. They provide money for personal loans, home improvements, back to work loans, working capital, bridging loans, property and equipment purchase, start up capital and business purchase.

Overdraft

Overdrafts are like a 'safety net' on your current account; they allow you to borrow up to a certain limit when there's no money in your account and can be useful to cover short term cash flow problems. Overdrafts offer more flexible borrowing than taking out a loan because you can repay them when it suits you, but they're not usually suitable for borrowing large amounts over a long period as the interest rate is generally higher than with a personal loan. You need a bank account in order to have an overdraft.

Buying on credit

Buying on credit is a form of borrowing. It can include paying for goods or services using credit cards or under some other credit agreement.

a Advantages and disadvantages of borrowing

The interest paid up on borrowed money is tax deductible. Therefore, cheaper Birr is paid back. Terms and conditions of borrowing are fixed and are subject to change in relation to changes in market conditions like price increments. As a result, costs will decrease and the value of the firm will increase.

The disadvantage of borrowing is that, if prices in the money market are going down, the borrower will be obliged to pay much more money as interest on fund borrowed. This is because terms and conditions are fixed. Bond indentures are burdensome due to inflexibility. In addition to this increase in debt may cause bankruptcy.

b Source of loan

The main sources of loan are saving institutions like commercial banks, saving and loan associations and credit unions. Others include consumer finance companies, insurance companies and private companies.

Group Work 11.6

Consider a company that need money to cover a Birr 1,000,000 credit.

Discuss the following two situations to set the credit.

- **a** Borrowing money from a bank.
- **b** Using overdraft facility from a bank.

11.4 TAXATION

Group Work 11.7

Discuss in small groups and present your findings to the whole class.

1. Why do governments collect taxes?
2. List out the different types of taxation.

As governments have played a growing role in all economies, they have used increasing amounts of resources for their activities, and taxes have constituted increasing percentages of national income. Either directly or indirectly, the various levels of government provide most education and pay a major proportion of medical bills. They provide national defence, police and fire protection and provide or support a substantial amount of housing, recreation facilities and parklands. They set health standards and ensure adequate water supplies, transportation and other public facilities. They seek to attain a distribution of income regarded as equitable, to stabilize the economy from periods of excessive inflation or unemployment, and to ensure an adequate rate of growth.

According to Richard Musgrave, governmental activities are divided into three parts.

1. **Allocation:** the activities involving the provision of various governmental services to society and thus involving the allocation of resources to the production of these services. Some of the services are strict public goods (e.g. national defence) some are ones involving externalities (e.g. education) some are provided by government to avoid private monopoly and costs of collection of charges (e.g. highways).

2 **Distribution:** the activities involving in the redistribution of income welfare programs, progressive tax structures and so forth.

3 **Stabilization and growth:** the activities designed to increase economic stability by lessening unemployment and inflation and influencing, if thought desirable, the rate of economic growth.

ACTIVITY 11.10

In order to do all the above mentioned activities, where does the government get money.

A Objectives of Taxation

Governments impose and collect taxes to raise revenue. Revenue generation however is not the only objective of taxation, though it is clearly the prime objective. Taxes as a fiscal policy instrument are used to address several other objectives such as:

1 **Removal of inequalities in income and wealth:** Government adopts progressive tax system and stressed on canon of equality to remove inequalities in income and wealth of the people.

2 **Ensuring economic stability:** Taxation affects the general level of consumption and production; hence it can be used as an effective tool for achieving economic stability. Governments use taxation to control inflation and deflation.

3 **Changing people's behaviors:** Though taxes are imposed for collecting revenue to meet public expenditure, certain taxes are imposed to achieve social objectives for example, to discourage consumption of harmful products, governments impose heavy taxes on production of tobacco and alcohol.

4 **Beneficial diversion of resources:** Governments impose heavy tax on non-essential and luxury goods to discourage producers of such goods and give tax rate reduction or exemption on most essential goods. This diverts producer's attention and enables the country to utilize limited resources for production of essential goods only.

5 **Promoting economic growth:** Economic growth depends on the generation of income from industrial agricultural and other areas. The rate of economic development goes up if more investment is available to all sectors. Tax policy of a government is a key element in planning the economic growth of a country.

B Principles of taxation

The compulsory payment by individuals and companies to the state is called **taxation**. A government imposes taxes to raise revenue to cover the cost of administration, the maintenance of law and order, defense, education, housing, health, pensions, family allowances etc. Now, the government has started to subsidize farming, industries, etc. In all these, taxes are imposed to provide revenue to cover government expenditure.

Adam Smith's Cannon of Taxation: Adam Smith has laid down principles or cannon of taxation in his book "Wealth and Nations". These cannons still constitute the foundation of all discussions on the principles of taxation.

To create an excellent system of taxation, it is necessary to first establish a set of high standard principles for taxation. Little or no attention has been paid by governing bodies to establish such important principles.

Group Work 11.8

Read a literature that can help to establish important principles that create good taxation system.

C Classification of taxes

ACTIVITY 11.11

Name some types of taxes you know.

In Ethiopia taxes are classified on the basis of impact (immediate burden) and incidence (ultimate burden) of tax. Taxes are classified into two broad categories. **Direct** and **Indirect taxes**.

1 Direct taxes

Direct tax is one in which the payer himself is the ultimate sufferer of its consequence. This means the incidence cannot be transferred to a third party. Direct taxes according to the Ethiopian tax law include all income taxes such as employment income tax, business income tax and land use fee, mining income tax and other income taxes. Generally direct taxes are income based taxes.

Schedules of income

Recently, Ethiopia has launched a tax reform program to achieve the objectives of democracy by considering taxation as one of the most important areas where attention is required. It resulted in the outcome of many important proclamations. The present income tax proclamation (No 286/2002) proclaimed after the tax reform program of the country, incorporated a number of tax bases as part of the development activities of the government.

The government has identified many tax bases for direct taxes. These tax bases are categorized into different schedules according to their nature in the proclamation. Thus the four schedules incorporated in direct taxes are schedules 'A' 'B' 'C' and 'D'. The bases for these schedules are.

Schedule A: Income from employment

Schedule B: Income from rental of building

Schedule C: Income from business

Schedule D: Other incomes which include royalties' income from technical services rendered outside the country, income from games of chance, dividend income, causal rental of property, interest income and gains from transfer of investment property.

Schedule A: Employment income tax

The employer assesses employment income tax. The tax is deducted at source before paying the monthly salary. For assessment of tax the employers make use of the following tax rates.

Taxable monthly income (birr)	Tax rate	Amount of tax (in birr)
UP to birr 150	Nil	Nil
151-650	10%	T × 10%-15.00
651-1400	15%	T × 15%-47.50
1401-2350	20%	T × 20%-117.50
2351-3550	25%	T × 25%-235.00
3551-5000	30%	T × 30%-412.50
More than 5000	35%	T × 35%-662.50

Example 1 Assume Ato Dagim earns a monthly salary of Birr 1350. His income tax will be calculated as follows.

Total taxable income	1350	
Less: the minimum amount not taxed	150	
Remaining taxable income	1200	
Less: first Birr 500 taxed at 10%		500 ×10% = 50.00
Remaining taxable income Birr 700 taxed at 15%		700 × 15% = 105.00
Total tax of the month		155.00

Ato Dagim's net income is then 1350 – 155 = Birr 1195.

ACTIVITY 11.12

If Ato Dagim whose salary was Birr 1350 got a salary increment of Birr 500,

a calculate the tax on the new increment.

b what will be his net salary after the increment?

Schedule B: Rental income tax

When leasing a building, certain items of expenses (deductible expenses) can be subtracted from the gross income in order to arrive at the amount that is taxed. The expenses allowable against the rental income are those incurred wholly or exclusively in connection with the leasing activity. Deductions include taxes paid with respect to the land and building leased except income taxes and a total of an allowance of 20% of the gross rent received; for repairs, maintenance and depreciation of such building, furniture and equipment. The tax rate for a body is 30% and others are as in the following schedule.

(T) Annual taxable income (birr)	Rate	Short cut formula
Upto-1800	Nil	Nil
1801-7800	10%	$T \times 10\%-180$
7801-16,800	15%	$T \times 15\%-570$
16,801-28,200	20%	$T \times 20\%-1410$
28,201-42,600	25%	$T \times 25\%-2820$
42,601-60,000	30%	$T \times 30\%-4950$
60,001-and above	35%	$T \times 35\%-7950$

Schedule C: Business income tax

The income tax proclamation (№ 286/2002) provides the tax rates that should be used for this purpose. The tax rate is applied on the assessed taxable income of the business unit. Once the declaration is made by the business unit, its accuracy is checked by the tax office through a process called tax assessment. Tax assessment is a tax review by a tax official of a tax declaration and information provided by a taxpayer and a verification of the arithmetical and financial accuracy of the declared tax liability. The procedure for the assessment of business income tax takes two forms.

- ✓ Assessment by books of accounts and
- ✓ Assessment by estimation.

Tax of those taxpayers who have different sources of income under schedule "C" will be assessed on the aggregate of all income.

The tax rates used for computation of income under schedule "C" are the same as that of schedule "B". Under schedule "C" there are three categories "a", b" and "**c**" of which a and b categories are assessed by books where as category "c" is assessed by estimation.

Schedule D: Other income taxes

People often get income from other sources in addition to (or other than) the income obtained from their employment, their business activities or their renting activities. The income from other activities is taxed at a flat rate as described below.

Source of income	rate
Royalty	5%
Technical services	10%
Dividend	10%
Interest	5%
Game of chance	15%
Casual rental of property	15%
Gain on transfer of investment property:	Gain on share capital 30%
	Other capital gain 15%

Example 2 Ato Tekle leased his personal car for two months at Birr 6000 per month. Such income is referred to as casual rental income by the tax expert.

i How much is the tax to be paid? **ii** Who is liable to pay the tax?

Solution

i Tax on casual rental of property = 15% of gross rental income

= 15% × (6000×2)

= 15%× 12,000.00 = Birr 1800.00

ii The receiver of the income, Ato Tekle, is liable to withheld and pays the required tax to tax authority.

Example 3 Selam owned 200,000 shares of common stock of Nile Company. The company declared and paid a dividend of Birr 2 per share.

i How much dividend is Selam entitled to?

ii How much is the tax to be paid?

Solution

i Dividend income = 200,000 shares × Birr 2 per share = Birr 400, 000.

ii Tax on Dividend income = 10%× 400,000 = Birr 40,000.

✍Note:

The dividend income after tax is paid = 400,000 – 40,000 = 360,000 and Nile Company is liable to pay the income tax to the tax authority.

Example 4 Ato Alemu has a deposit with Awash Bank on which he is eligible to get interest Birr 140,000 in a year. How much of this is withheld by Awash Bank for tax purpose?

Solution Tax withheld = 140,000 × 5% = Birr 7,000

Example 5 Fitsum won Birr 300.000 from the National Lottery Administration. Tax is paid only if the amount exceeds Birr 100.

Required:

a What is the amount of tax withheld by the lottery authority?

b How much did Fitsum receive?

Solution

a Tax with held = 300,000 × 15%= 45,000

b Amount received by Fitsum = 300,000.00 - 45,000.00 = Birr 255,000.00

Example 6 The author of a book gave the copy right of the book to Mega Publishers, Ethiopia, for royalty of Birr 280,000 How much tax will Mega Publishers withhold on this royalty payment?

Solution Royalty = 280,000 × 5% = Birr 14,000

Example 7 Ato Samuel acquired 1000 shares of Admas Co. for Birr 4,500 each and sold them at Birr 6,000 each. How much does he pay as capital gain tax?

Solution Gain = (6000 – 4500) × 1000 = Birr 1,500,000.

Capital gain tax = 1,500,000.00 × 30% = Birr 450,000.

Example 8 Kurtu Trading Co. sold one of its building for Birr 980,000 which it acquired for 720,000. Compute the capital gain tax.

Solution Capital gain= 980,000 – 720,000 = Birr 260,000.

Capital gain tax = 260,000 × 15% = Birr 39,000.

2 Indirect taxes

Indirect tax is a tax in which the burden may not necessarily be swallowed by business; which means, indirect taxes can be shifted onto other persons. Generally the tax incidence of indirect tax is on the ultimate consumer; however, sometimes a seller might absorb such indirect tax to be competitive in the market. This action reduces its profit. Indirect taxes are consumption based taxes. In Ethiopia the indirect tax category includes, value added tax (VAT), excise tax, turnover tax (TOT), custom duties and stamp duty.

Value Added Tax (VAT)

VAT is a levy imposed on business at all levels of production and distribution of goods and services. It is determined on the basis of the increase in price, or value, provided by each stage in the chain of distribution. It is a general consumption tax assessed on the value added to goods and services. Some goods are exempted from VAT. Supplies which are not exempted are called taxable supplies. Taxable supplies and import goods are taxed at a flat rate of 15% in our country. Some taxable supplies are zero rated. Zero rated supplies are those on which VAT on supply/sale is charged at zero rates.

In Ethiopia invoice credit method is used for collection of VAT. Under this method, VAT payable is the difference between the tax charged on taxable transactions and tax paid on import of goods or on the purchase of supplies where such supplies are used or to be used for the taxable transactions.

Example 9 Nokia Company purchased Mobiles for Birr 54,000. net and will be invoiced and will pay the supplier Birr 62,100 of which 8,100 is VAT. Nokia sell these mobiles for 86,250 (Birr 75,000 + Birr 11,250 VAT.). The VAT liability of Nokia company is Birr 3,150 (11,250 – 8,100).

The detail is illustrated below.

Purchase and sale of Mobile			
	Birr	**VAT (15%)**	**Explanation**
Revenue	75,000.00	11250	Output tax
Cost	54,000.00	8100	Input tax
Value added	21,000.00	3150	VAT liability.

Turnover Tax (TOT)

To enhance fairness in commercial dealings and to make a complete coverage of the tax system, a turnover tax is imposed on those persons who are not required to register for VAT, but supply goods and services in the country. As a result, persons who are engaged in the supply of goods and rendering of service (which are taxable) and who are not required to register for VAT have to pay turnover tax on the value of goods they supply or on the value of services they render. TOT is computed as per the proclamation No 308/2002. The TOT rate is

- ✓ On goods sold locally: 2%
- ✓ On services rendered locally:
  - Contractors, grain mills, tractors, and combine harvesters: 2%
  - Others:10%

Example10 Elsa Stationery has daily sales of Birr 205. The stationery is open 280 days per year. How much is the turnover tax payable by Elsa?

Solution Annual sales = 280×205 = Birr 57,400.

TOT = Birr 57,400 × 2% = Birr 1148.

Excise Tax

With a view to increase the revenue of the government to provide for public goods and services and to reduce the consumption of specific goods, the governments of Ethiopia levies excise tax on selected items of goods that are supplied in the country. As per the excise tax proclamation No 307/2002, the items of goods that are subject to excise tax in Ethiopia are: goods imported to the country and goods produced locally. The tax is imposed equally on both imported and locally produced goods at a rate defined in the proclamation. The major items included are sugar, salt, tobacco, alcohol, textile, fuel, jewelers, vehicles and televisions.

Example 11 Awassa Textile incurred the following costs during the first month of 2002 E.C for textile production. Compute the excise tax payable.

Material used	Birr 1,506,000
Direct labour	Birr 404,000
Indirect costs	Birr 900,000
Total	Birr 2,810,000

(Note:- Textile is taxed at a rate of 10%)

Solution Excise tax payable = 2,810,000 ×10% = Birr 281,000

Example12 A company is importing sugar from China incurring costs of Birr 842,000 Birr 210,500 and Birr 165,500 for purchasing, insurance and freight respectively. Compute the excise tax payable.

(Note:- Sugar is taxed at a rate of 33%)

Solution Total cost (purchase, insurance and freight)

(842,000 + 210,500 + 165,500) = Birr 1,218,000.

Excise tax payable = 1,218,000 × 33% = Birr 401,940.

Customs duty

Customs duty refers to the tax tariff imposed by the tax authority directly on the activities of import and export of goods and services. Custom duty is levied on cost of production for locally produced goods and cost, insurance and freight (CIF) paid for imported items. Duties of customs are levied on goods imported to or exported from Ethiopia at a rate ranging from 0 to 35% as follows.

Imports	Tax rate (%)
Raw materials, capital goods, chemicals and pharmaceuticals	0-20
Durable and non Durable consumer goods	20-35
Luxuries and goods that can be produced locally	30-35

Items like diplomatic and consular missions, personal effects, grants and gifts to Ethiopia, fire fighting instruments and appliances, trade samples, defence and public security equipments, materials for handicapped and similar items are exempted from customs duty.

Example 13 Kent tobacco importing company paid cost of purchase \$ 120,000.00, insurance premium and freight costs are, respectively, \$12,000.00 and \$8,000.00.The exchange rate is currently \$1=12.50 Birr.

Compute the customs duty. *(Note:- Tobacco is taxed at 35%.)*

Solution CIF = (120,000 + 12,000 + 8,000) × Birr 12.50

= 140,000 × Birr 12.50 = Birr 1,750,000.

Custom Duty = Birr 1,750,000 ×35% = Birr 612,500.

Exercise 11.9

1 Find the income tax of the following employees of Nyala Insurance sh. Company.

- **a** W/ro Mebrat with monthly salary of Birr 850.
- **b** Ato Tesfu with monthly salary of Birr 2,390.
- **c** Dr. Gebru with monthly salary of Birr 5,400.

2 Buna Bank declared to pay 20% dividend to its share holders. Find the dividend earned and tax to be paid by the following share holders.

- **a** Mesfin with Birr 300,000 worth of shares.
- **b** Askale with Birr 100,000 worth of shares.
- **c** W/ro Almaz with Birr 450,000 worth of shares.

3 If Kassa won a lottery worth of Birr 150,000, find the amount of tax he is liable to and his net income.

4 Zewdinesh rented her loader for 10 days to a contractor at a rate of Birr 5,000 per day. Determine the amount she earns after tax.

5 A company purchased the following items from a stationary.

Item	Quantity	Unit price before VAT	Total price
Computers	5	12,500	
Toners	5	2,400	
Cables	5	150	

i Complete the table

ii What is the total VAT to be paid?

iii What is the total price of the items including VAT?

iv If the company want to pay for the stationary after subtracting a 2% withholding tax before VAT,

a what is the amount that will be subtracted by withholding?

b what is the amount that the company has to pay for the stationary?

6 A company wants to buy five cars from MOENCO. If the price of each car including VAT is Birr 550,000, then

i what is the total price of each car before VAT?

ii if the company wants to subtract a 2% withholding tax before VAT, what is the amount to be subtracted?

iii what is the amount that the company should pay to MOENCO after withholding 2% is subtracted?

7 A shoe dealer purchased net Birr 8,000 worth of shoes from a shoe company.

a Find the amount it is to pay the company including VAT.

b If the dealer sold the shoes for Birr 12,000, find the amount of VAT liable to the dealer.

8 An artist sold his new song to a producer company for Birr 350,000. What is the royalty that should be paid by the artist.

Key Terms

annually	jointly proportional	rate
base	liquidity	ratio
book value	markup	reducing-balance method
commercial discount	mean proportion	restrictions
compound interest	ordinary annuity	safety
compound proportion	percentage	salvage value/residual value
depreciation	present value	semi-annually
earnings	principal	simple interest
fixed-installment method	proportion	simple proportion
future value	proportionality constant	taxes
interest	quarterly	terms

Summary

1 A **ratio** is a comparison of two or more quantities expressed in the same unit of measurement.

2 A **rate** is a comparison of two or more quantities expressed in different unit of measurement.

3 A ratio can be a rate.

4 $\text{Rate of change} = \dfrac{\text{amount of change}}{\text{original amount}} = \dfrac{\text{final amount - original amount}}{\text{original amount}}$

5 A **proportion** is a statement of equality between two ratios.

6 A **compound proportion** is a situation in which one variable quantity depends on two or more other variable quantities.

7 A **percentage** is the numerator of a fraction whose denominator is 100.

8 **Percentage = base × rate**

9 **Markup = Selling price - Cost**

10 The **future value of a simple interest** investment is obtained by

$A = P + I = P + Prt = P\,(1 + rt)$

11 The **future value of a compound interest** investment is obtained by

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

12 The **future value of an ordinary annuity** is given by $S = R\left(\dfrac{(1+i)^n - 1}{i}\right)$ and the amount of interest is $I = S - nR$.

13 **Plant assets** or **fixed assets** are tangible assets used in business that are of a permanent or relatively fixed nature.

14 **Depreciation** of a plant asset is decrease in usefulness of the asset.

Review Exercises on Unit 11

1 What is the ratio of 1.8 km to 800 meter?

2 In a family there are three daughters and a son. What is the ratio of the number of

a females to the number of people in the family?

b males to the number of females in the family?

3 Allocate a profit of Birr 21,400 of a company among three partners in the ratio of their share of the company $\frac{1}{3}:\frac{2}{5}:\frac{2}{7}$.

4 15 workers can accomplish a job in 28 days. At the same rate by how many workers can the work be accomplish in 8 days less time?

5 What percent of Birr 52 is Birr 3.12?

6 8.35% of what amount is Birr 18.37?

7 A 6% tax on a pair of shoes amounts to Birr 10.20. What is the cost of the pair of shoes?

8 If the average daily wage of a labourer increased from Birr 16.00 to Birr 21.64 in the last three years, what is the rate of increase?

9 A radio recorder sold for Birr 210 has a markup of 25% on the selling price. What is the cost?

10 Ato Alula deposited Birr 3,000 in a saving account that pays 6% interest rate per year, compounded quarterly. What is the amount of interest obtained at the end of seven years? (No deposit or withdrawal is made in these seven years)

11 Ato Alemu makes regular deposits of Birr 230 at the end of each month for 3 years. What is the future value of his deposit, if interest rate per year is 9% compounded monthly? What is the amount of interest?

12 At the end of each month Ato Mohammed deposits 10% of his salary in a saving institution that pays annual interest rate of 6% for one year and then 15% for the next 3 years. If the salary of Ato Mohammed is Birr 1800, find the future value of his deposits at the end of the 4 years.

13 A piece of machinery costs Birr 50,000 with an estimated residual value of Birr 7,000 and a useful life of 8 years. It was placed in service on April 1 of the current fiscal year. Determine the accumulated depreciation and book value at the end of the following fiscal year using:

a the fixed installment method. **b** the double reducing balance method.

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